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FOUNDATIONS

***Tarski, Alfred. Introduction to Logic and to the Methodology of Deductive Sciences.** Oxford University Press, New York, 1941. xviii+239 pp. $2.75.

This is a translation and revision, with a considerable amount of added material, of a book published in Polish in 1936 and in German in 1937. Besides being a clear and readable exposition of the elements of modern logic, it is so arranged that it may be used as a textbook in an introductory undergraduate course in logic and axiomatics. Features not present in the previous edition include the introduction of logical symbols, the truth table method, the calculus of relations and many new exercises.

The book starts with a discussion of variables, sentential functions and quantifiers, followed by a short description of the calculus of elementary propositions, the truth table method and rules of inference. The next three chapters deal with the theory of identity, the calculus of classes and the calculus of relations, including the notions of cardinal number, function and one-to-one correspondence. The sixth chapter, on the deductive method, is perhaps the most significant in the book. It introduces the notion of an axiom system and of a formalized deductive theory, and such concepts as consistency and completeness of systems. The author presents here his view of the importance of methodology (metamathematics) as an independent subject. The last four chapters should be of particular interest to mathematicians. The ideas and methods of the first part of the book are illustrated here by applying them to the actual construction of mathematical theories based on various axiom systems. Four different postulate systems for ordered abelian groups are compared, and finally two different systems of postulates for the real numbers are considered.

The book itself is not a formalized deductive theory of logic, but rather, as its title indicates, an introduction to logic. A good feature is the large collection of exercises, many of them from the field of mathematics. There seems to be no good reason for the author's use of the terms sentential function and sentential calculus in place of the well established terms propositional function and propositional calculus. Although emphasis is placed on the view of Leśniewski that definitions are not mere abbreviations but must conform to rules of definition, this point is not made very clear. In general, however, the book is a model of clarity. It fills the need for an easily intelligible and up-to-date treatment in English of those topics in logic which are of greatest importance for mathematics.

O. Frink (State College, Pa.).

McKinsey, J. C. C. A correction to Lewis and Langford's Symbolic Logic. J. Symbolic Logic 5, 149 (1940).

Lewis and Langford state in their Symbolic Logic that, if a formula T, which involves only the element p, is provable in the two-valued Łukasiewicz-Tarski calculus, then the formulas $p \supset T$ and $(\sim p) \supset T$ are provable in the three-valued calculus. It is shown by a counterexample that this statement is incorrect. *O. Frink* (State College, Pa.).

Dugundji, James. Note on a property of matrices for Lewis and Langford's calculi of propositions. J. Symbolic Logic 5, 150–151 (1940).

It is shown that for none of Lewis' systems of strict implication S1–S5 is there a realization, with only a finite number of elements, in which those and only those formulas hold which are provable in the system in question. The same result is obtained for the system arising from B1–B8 and $\Diamond \Diamond p$. The method used is that introduced by Gödel in proving the corresponding theorem for the Heyting calculus.

O. Frink (State College, Pa.).

Fitch, Frederic B. Closure and Quine's *101. J. Symbolic Logic 6, 18–22 (1941).

W. V. Quine in his Mathematical Logic adopts a definition of closure for statements. He states [p. 79]: "If φ has n free variables, there are $n!$ statements which can be formed by prefixing distinct quantifiers in one or another order corresponding to all the free variables; though all $n!$ statements are, as just remarked, equivalent. An arbitrary one of these statements will be referred to as the closure of φ; the one, say, in which the added quantifiers are arranged in alphabetic order." Later [p. 88] he offers the following: *101. $\vdash \ulcorner(\alpha)(\beta)\varphi \supset (\beta)(\alpha)\varphi\urcorner$. The present paper proposes to eliminate this last mentioned principle by allowing (as one alternative) that each formula has "not merely one closure but as many closures as can be obtained by permuting in various ways the n prefixed universal quantifiers. In this way alphabetic order becomes irrelevant." A second alternative redefines closure inductively in terms of semiclosure: "The semiclosure of f is $(a)f$, if a is free in f and is the left-most variable in f. If there is no variable free in f (so that f is not a matrix but a statement) then the semiclosure of f is f itself." The closure of a matrix is defined thus: "If f is the closure of $(a)g$, and if $(a)g$ is the semiclosure of g, then f is the closure of g. The details are presented completely. *A. A. Bennett* (Providence, R. I.)

Berry, George D. W. On Quine's axioms of quantification. J. Symbolic Logic 6, 23–27 (1941).

This paper deals with the same question as the paper reviewed above and proposes the elimination of *101 by means of an "otherwise trivial shift in meaning" of Quine's other principles of quantification: *100, *102, *103, *104. The device is to employ as "alphabetic" order, not, as with Quine, $w, x, y, z, w', x', y', z', w'', \cdots$, with a first but no last variable, but, instead, the order $\cdots, z'', w', x', y', z', w, x, y, z$, with a last but no first variable. The proofs are offered in detail. *A. A. Bennett* (Providence, R. I.)

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Hosiasson-Lindenbaum, Janina. On confirmation. J. Symbolic Logic **5**, 133–148 (1940).

Let a, b, c be variable names of sentences belonging to a certain class, with $a+b$, ab, $\bar{a}$, $a \supset b$, $a=b$ denoting respectively "a or b," "a and b," "not-a," "b is a consequence of a," "a is equivalent to b." Further, let $C(a, b)$ be a real non-negative function of a, b, where b is self-consistent, and let $C(a, b)$ be read as "the degree of confirmation of a with respect to b." Then the following propositions hold: (1) if $b \supset a$, $C(a, b)=1$; (2) if $c \supset \overline{ab}$, $C(a+b, c)=C(a, c)+C(b, c)$; (3) $C(ab, c)=C(a, c)\cdot C(b, ac)$; (4) if $b=c$, $C(a, b)=C(a, c)$. The author uses (1)–(4) as axioms for a theory of confirmation by means of which she solves rigorously a variety of problems on confirmation. The axioms parallel St. Mazurkiewicz's axioms for probabilities. Among the author's applications of her theory are the solution of a paradox due to C. G. Hempel, a discussion of the nature of "metaphysical" hypotheses, and an answer to the question "whether the C of a fact α_2 is increased when there is observed another fact α_1 which is a consequence of the same hypothesis b as the first fact but which is not certain a priori."

B. A. Bernstein (Berkeley, Calif.).

Skolem, Th. Einfacher Beweis der Unmöglichkeit eines allgemeinen Lösungsverfahrens für arithmetische Probleme. Norske Vid. Selsk. Forh. **13**, 1–4 (1940). [MF 3865]

An arithmetic proposition is defined to be a propositional function of integers which can be constructed by means of logical symbols from the relations $x=y$, $x=y+z$, $x=yz$. Using results of Gödel and Kleene, it is shown that every equation between general recursive functions is equivalent to an arithmetic proposition. The set of all arithmetic propositions is effectively enumerable. Let $C_m(n)$ be the mth arithmetic proposition in some enumeration. If there were a general method of solving in a finite number of steps all arithmetic problems, there would exist a general recursive function $f(m, n)$ assuming the values 1 and 0 depending on whether $C_m(n)$ is true or false. By the preceding result, the equation $f(n, n)=0$ would be equivalent to some arithmetic proposition $C_k(n)$. This implies a contradiction, however, since $C_k(k)$ would then be equivalent to its own negation. Hence no general method of solving all arithmetic problems exists.

O. Frink (State College, Pa.).

Bernays, Paul. A system of axiomatic set theory. Part II. J. Symbolic Logic **6**, 1–17 (1941).

This is the second installment of a system of axiomatic set theory of which the first installment appeared in the J. Symbolic Logic **2**, 65–77 (1937). The system is a modification of the von Neumann system [Math. Z. **27**, 669–752 (1928)] with principal features as follows: there are two primitive categories, class and set, and two relations of membership, one between sets and classes and one between sets and sets; the system is elementary in the sense that it can be formalized on the basis of the logical calculus of the first order ("engerer Prädikatenkalkül") with equality and descriptions; the notions of ordered pair and function are defined; there are no individuals which are not sets. In the first part the author introduced the more basic axioms, namely, the axioms of definiteness (Bestimmtheit), the existence of the null set and of the set obtained by adjoining a set as a new element to a given set and the axioms for the construction of classes. The principal result of this first part was the "class theorem," showing that the axioms are sufficient for the existence of the class of n-tuples corresponding to any n-ary predicate constructed by the logical calculus under certain restrictions. In the present paper the author introduces the axioms of a more properly set theoretic nature: the axiom of choice; the axioms of subsets (Aussonderung), replacement, sum and power; the axiom of infinity; and the restrictive axiom (Axiom der Fundierung). He derives some immediate consequences and alternative forms notably of the axiom of choice and that of replacement. He then develops the theory of ordinals using a direct definition like that of R. M. Robinson [J. Symbolic Logic **2**, 29–36 (1937)], which does not relate them to a theory of order in general. On the basis of this a theory of finite ordinals and finite sets is developed sufficiently to form a foundation for the theory of numbers, including recursive definitions and the notion of the least number satisfying a condition. This theory of ordinals, and hence the theory of finite numbers, is developed using only the axioms of Part I and in addition either the restrictive axiom or the axiom of subsets. Further results bearing on analysis are promised for later installments. [A detail of some importance is the reduction of the general recursive definition to an iterative one.]

H. B. Curry (State College, Pa.).

Fraenkel, Abraham Adolf. Natural numbers as ordinals. Scripta Math. **7**, 9–20 (1940). [MF 4128]

Vredenduin, P. G. J. Logistic. Chr. Huygens **18**, 170–211 (1940). (Dutch) [MF 4357]

Bieberbach, Ludwig. Die völkische Verwurzelung der Wissenschaft (Typen mathematischen Schaffens). S.-B. Heidelberger Akad. Wiss. **1940**, no. 5, 31 pp. (1940). [MF 4123]

***Hardy, G. H. A Mathematician's Apology.** Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1940. vii+93 pp. $1.00.

"I shall ask, then, why is it really worth while to make a serious study of mathematics? What is the proper justification of a mathematician's life? And my answers will be, for the most part, such as are to be expected from a mathematician. . . . But I should say at once that, in defending mathematics, I shall be defending myself, and that my apology is bound to be to some extent egotistical." [Quotation from p. 5.]

G. Pólya (Providence, R. I.).

THEORY OF GROUPS

Miller, G. A. Maximal subgroups of a given group. Proc. Nat. Acad. Sci. U. S. A. **27**, 68–71 (1941). [MF 3656]

The author describes all groups which separately contain one, two or three maximal subgroups. Those with one must be cyclic of order p^m; those with two must be the direct product of two such cyclic p-groups with different primes; those with three must be either the direct product of three such cyclic p-groups with distinct primes, or must be of order 2^m. In any case if a group G has just three maximal subgroups they must be invariant. It is pointed out that a necessary and sufficient condition that a permutation group be primitive is that the subgroup composed of all its permutations which omit a given letter be maximal.

J. S. Frame (Providence, R. I.).

Iwasawa, Kenkiti. Ueber die Struktur der endlichen Gruppen, deren echte Untergruppen sämtlich nilpotent sind. Proc. Phys.-Math. Soc. Japan (3) **23**, 1–4 (1941). [MF 3906]

G. A. Miller and H. C. Moreno [Trans. Amer. Math. Soc. **4**, 398–404 (1903)] have characterized the finite non-abelian groups without non-abelian proper subgroups. Extending these results the author proves that the finite group G is not nilpotent though all of its proper subgroups are nilpotent if, and only if, the following conditions are satisfied. (1) G contains a normal subgroup Q of order a power of a prime q such that G/Q is cyclic of order a power of a prime $p \neq q$. (2) If Q' is the commutator subgroup of Q, then G/Q' is non-abelian, though each of the proper subgroups of G/Q' is abelian. (3) G^p as well as PQ' for P a p-Sylow subgroup of G are nilpotent. *R. Baer.*

Hall, P. The classification of prime-power groups. J. Reine Angew. Math. **182**, 130–141 (1940). [MF 3426]

In order to facilitate the classification of groups, especially of prime power groups, the author introduces an equivalence relation $\sim$ called isoclinism, which classifies all groups in families of isoclinic groups. If Z_1 and Z_1' denote the centrals of two groups G and G', if H_2 and H_2' denote their commutator subgroups, and if the relation of isomorphism is denoted by $\cong$, then by definition $G \sim G'$ if and only if it is possible to find isomorphisms (1) $G/Z_1 \cong G'/Z_1'$ and (2) $H_2 \cong H_2'$ so that, whenever elements $Z_1 a$ and $Z_1 b$ of G/Z_1 correspond to elements $Z_1' a'$ and $Z_1' b'$ of G'/Z_1', then the commutator $[a, b]$ of H_2 shall correspond to $[a', b']$ of H_2'. All Abelian groups G satisfy $G \sim 1$, and constitute the first family of groups. Some fundamental facts about isoclinism are contained in the theorems: (1) G is isoclinic with its subgroup K if and (for finite G/Z_1) only if $KZ_1 = G$; (2) G is isoclinic with its quotient group G/K if and (for finite H_2) only if $K \wedge H_2 = 1$; (3) in every family there exist groups for which $Z_1 \leqq H_2$: these are called stem groups of the family. The usefulness of isoclinism in the study of the comparative anatomy of prime power groups is illustrated by two examples: (a) the classification of groups of order p^5, $p > 3$, and (b) the classification of p-groups having an Abelian subgroup of index p. *J. S. Frame.*

Hua, Loo-keng and Tuan, Hsio-fu. Some "Anzahl" theorems for groups of prime-power orders. J. Chinese Math. Soc. **2**, 313–319 (1940). [MF 4216]

If G is a group of order a power of a prime $p \neq 2$, then denote by $p^{n(G)}$ the order of G and by $p^{n(G)-d(G)}$ the maximum-order of the elements in G. Making use of the enumeration-principle of Ph. Hall [Proc. London Math. Soc. (2) **36**, 29–95 (1933)] the authors prove: If $5 \leqq n(G)$, $2 \leqq d(G)$, then the number of subgroups S such that $2 < n(S) < n(G) - 1$, $d(S) = 0$ (that is, S is cyclic) is $\equiv 0$ modulo p^2; and the number of subgroups S such that $3 < n(S) < n(G)$, $d(S) = 1$ is $\equiv 0$ modulo p. *R. Baer.*

Schmidt, O. J. Groups with two classes of non-invariant subgroups. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 291–309. (Russian) [MF 3522]

Finite groups having precisely two classes of non-invariant subgroups are completely catalogued. There are eight classes of these groups; one consists of the tetrahedral group, another of a group of order eight, a third of a group of order sixteen. Each of the remaining is an infinite class. All but one of these consists of groups which are not the direct product of their Sylow subgroups, and of these only one contains groups whose orders are divisible by more than two primes (and then, necessarily, of the form $p^n qr$). The groups in a given class have the same number of generators (two or three, always) and the same general form of defining relations. These are completely set forth. Vital to the analysis are the author's earlier classification of groups having just one class of non-invariant subgroups [in Russian, Rec. Math. [Mat. Sbornik] **33**, 161–172 (1926)] and the following (Lemma): if P is a Sylow subgroup of G, N the normalizer of P, and if P lies in H, an invariant subgroup of G, then the index of $H \cdot N$ in N is equal to the index of N in G. *L. Zippin* (Flushing, N. Y.).

Speiser, Andreas. Gruppen aus der Klassenkörpertheorie. J. Reine Angew. Math. **182**, 178–179 (1940). [MF 3431]

The author makes a number of short remarks concerning applications of groups and their representations to class field theory. He first considers a basis for the units of an algebraic number field K which is normal over a subfield k. If a transformation of the Galois group G of K with regard to k is applied, each unit is transformed into a product of powers of the units. The exponents in these formulas furnish a representation of G by matrices with integral rational coefficients. This representation can be used for a study of the properties of the units. Next, a proof of the following group-theoretical statement is given: Let G be a group of finite order g and p a prime dividing g. If the p-Sylow-subgroup P of G is normal and abelian of type $(p, p, \cdots, p)$, then P contains a normal subgroup A of G such that G/A is a direct product of two factors the first of which has an order prime to p, while the second one is isomorphic with the intersection of P with the center of G. An application of this remark is given. The second section of the paper is concerned with the connection between the existence of splitting fields of given type for simple algebras and the existence of solutions of embedding problems for fields. It seems that all these results are contained in a paper of the reviewer [J. Reine Angew. Math. **168**, 44–64 (1932)]. In the last section, Speiser states that in certain cases the index of the class group corresponding to a relatively cyclic extension field of a given algebraic number field can be obtained by means of the formulas which describe the structure of the group of inertia; no details are given. *R. Brauer.*

Nielsen, Jakob. The symmetric and the alternating group. Mat. Tidsskr. B. **1940**, 7–18 (1940). (Danish) [MF 3877]

In a previous paper [Math. Ann. **91** (1923)] the author has proved that the symmetric group S_n can be generated by two generators P and Q satisfying (*) $P^2 = Q^n = (QP)^{n-1} = (PQ^{-i}PQ^i)^2 = 1$ for $2 \leqq i \leqq n/2$. In his investigations on knots, Artin [Abh. Math. Sem. Hansischen Univ. **4** (1926)] found that (*) can be replaced by the simpler relation (**) $P^2 = Q^{-n}(QP)^{n-1} = (PQ^{-i}PQ^i)^2 = 1$. The relation (*) was obtained also by Coxeter and Todd [Proc. Cambridge Philos. Soc. **32** (1936)] and (**) was subsequently proved by Coxeter [Proc. Cambridge Philos. Soc. **33** (1937)]. Now the author shows that (**) can also be proved by a slight modification of his original proof for (*), which is based on an inductive argument. *W. Feller* (Providence, R. I.).

Goheen, Harry. Proof of a theorem of Hall. Bull. Amer. Math. Soc. **47**, 143–144 (1941). [MF 3830]

The theorem is as follows. "If g is the order of a soluble group G and m is a divisor of g such that m and g/m are

relatively prime, then G has a subgroup of order m and furthermore all the subgroups of order m in G are conjugate under G." This is proved by induction over g, assuming the result for all soluble groups of orders less than g (and in particular for an invariant subgroup of prime index in G). *H. S. M. Coxeter* (Toronto, Ont.).

Lewis, F. A. On the group of isomorphisms of an Abelian group of order n^m and type $(1, 1, \cdots, 1)$. Amer. Math. Monthly **48**, 199–201 (1941). [MF 4136]

The general element of an Abelian group G of order n^m and type $(1, 1, \cdots, 1)$ can be represented in the form $S_1^{x_1}S_2^{x_2}\cdots S_m^{x_m}$, where $S_1, \cdots, S_m$ denote a fixed set of generators and the x_i vary independently from 0 to $n-1$. The order of the group of isomorphisms of this group is the number of distinct ways these generators can be selected, and is shown to equal

$$n^{m^2}\prod_{i=1}^{r}\prod_{i=1}^{m}(1-p_i^{-1}),$$

where $p_1, \cdots, p_r$ are the distinct prime factors of n. An isomorphism of G replaces S_i by $S_1^{a_{1i}}S_2^{a_{2i}}\cdots S_m^{a_{mi}}$, and is therefore given by a linear homogeneous transformation (mod n) on the exponents, which, since it has an inverse, must have its determinant $|a_{ij}|$ relatively prime to n. This general linear homogeneous group (mod n) is isomorphic with the group of isomorphisms of G. *J. S. Frame.*

Finan, E. J. On groups of subtraction and division. Amer. Math. Monthly **48**, 3–7 (1941). [MF 3716]

Continuing an investigation of G. A. Miller in a paper with the same title [Quart. J. Math. **1906**, 80–87], the author studies the group generated by two operators $R_1(x)=r/x$ and $R_2(x)=s-x$, of period two, where r and s are complex numbers. He derives formulas for $(R_2R_1)^k$, and conditions that the group generated by R_1 and R_2 should be of finite order $2n$. It is found that s^2/r must be real and must be a root of a certain algebraic equation $f_i(x)=0$. The properties of the equation $f_i(x)=0$ which are given would have been more easily obtained had it been noticed that, if R_2R_1 is of order n, then $s^2/r=4\cos^2(k\pi/n)$, where k is an integer relatively prime to n. *J. S. Frame.*

Sanov, I. N. Solution of Burnside's problem for exponent 4. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 166–170 (1940). (Russian) [MF 3312]

Burnside stated without proof that, if a group has a finite number of generators, and if each element A satisfies the equation $A^n=E$, then the group is finite. The author first proves the theorem for $n=4$, then for $n=3$, and finally for $n\leqq 4$. He calls a group G a Burnside group if it satisfies the hypotheses of the theorem. The argument proceeds by induction on the number e of generators. For $e=1$ the group is finite. Assume that a Burnside group is finite if it has $e-1$ generators of period 2 or 4, and consider a Burnside group having these and one additional generator of period 4. In one step adjoin the square of this generator; in the second step adjoin the generator itself. Then in each case the general element L of the larger group G can be written

$$L=P_1cP_2c\cdots cP_ncP_{n+1},$$

where P_i and c^2 are elements of a Burnside subgroup G_1 of G which is known to be finite; and it must be shown that G itself is finite. The method of proof is to show that, for any element L of G, the number of elements P_i from G which are required to express it in the above form can be reduced until $s\leqq\gamma_1$, where γ_1 is the order of G_1. Hence the order of G is less than $\gamma_1^{\gamma_1+1}$ and must be finite. The proof for $n=3$ and $n\leqq 4$ is similar. *J. S. Frame.*

Grün, O. Zusammenhang zwischen Potenzbildung und Kommutatorbildung. J. Reine Angew. Math. **182**, 158–177 (1940). [MF 3430]

If F_n is the free group with n generators, F_n^m the normal subgroup generated by all the mth powers of elements in F_n, and if $G(m, n)$ is the quotient-group F_n/F_n^m, then it is Burnside's problem whether or not $G(m, n)$ is finite. As long as this problem is unsolved, and in particular if the answer should be in the negative, one may consider the problem whether or not the orders of the finite quotient-groups of $G(m, n)$ are bounded. If m is a power of a prime number, and if G_i is the ith term in the lower (descending) central series of the group G, then the orders of the finite quotient-groups of $G(m, n)$ are bounded if, and only if, there exists some i such that $G_i(m, n)=G_{i+1}(m, n)$. The author is concerned with this problem; and as an application of his results he may prove that $G_7(5, 2)=G_8(5, 2)$. In order to obtain results of this type, the author investigates the problem of finding conditions assuring that certain elements are in G_{i+1}, though one could offhand only expect them to be in G_i. As a typical example of the author's many results in this direction we mention: If $m=p^j$, t an element in $G_i(m, n)$, and if (t, s, h) is defined inductively as the commutator of s and $(t, s, h-1)$, then $(t, s, up^j-(u-1)p^{j-1}-1)^{p^{j-a}}$ is an element in $G_{2i+1}(m, n)$ for $u=1, 2, \cdots, j$. The method employed is the study of the group-ring of $G(m, n)$, in particular application of the ideas and results of W. Magnus [Math. Ann. **111**, 259–280 (1935)] and Witt [J. Reine Angew. Math. **177**, 152–160 (1937)]. *R. Baer.*

Dietzmann, A. P. Über die Kongruenz der Systeme von Elementen einer Gruppe nach einem Doppelmodul. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 315–319 (1940). [MF 3554]

Subsets M and M_1 of a group G are said to be congruent with respect to a double modulus (H, Q) of two subgroups if the double cosets HMQ and HM_1Q are the same. The author writes $M\equiv M_1(H, Q)$ for this relation, which is clearly symmetric, reflexive and transitive. The principal result of the paper is that, if $MK\equiv M(H, Q)$ for an invariant complex K and $HMQ\neq G$ and either (1) $K=K^{-1}$ or (2) $M=M^{-1}$ and $Q=H$, then K generates a proper normal subgroup of G. This includes previous results using simple congruences by Dietzmann and Kulakov [the same C. R. (N.S.) **3**, 11–12 (1935)] and by Sčypanoff [the same C. R. (N.S.) **25**, 99–102 (1939); cf. these Rev. **1**, 258]. *M. Hall.*

Sčypanoff, P. Sur les diviseurs normaux d'un groupe. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 529–534 (1940). (Russian. French summary) [MF 4090]

In a group G a complex S is said to be represented on the complex S' modulo the semi-group H if for every s of S there is an h of H and an s' of S' such that $s=hs'$. It is shown that, if a normal complex A represents S on S', if A^{-1} (the inverses of A) represents S' on S, and if further $HS=G$, then G has a proper normal subgroup. *M. Hall.*

Rainville, Earl D. A discrete group arising in the study of differential operators. Amer. J. Math. **63**, 136–140 (1941). [MF 3637]

Multiplying functions by constants, by the variable and

differentiating them any number of times are linear operators which generate a ring R. The author discusses a group of linear transformations of R which he proves to be exactly the group generated by three elements b_1, b_2, b_3 subject to the relations: $b_1^4=b_2^2=(b_1b_2)^2=(b_1b_3)^2=1$, $(b_2b_3)^2=b_1^2$.

R. Baer (Urbana, Ill.).

Dehn, M. Ueber Abbildungen. Mat. Tidsskr. B. **1939**, 25–48 (1939). [MF 716]

The author considers transformations $z'=(az+b)/(cz+d)$ of a complex variable z with a, b, c, d real and $ad-bc>0$. When the two fixed points $z'=z$ are real and different, the transformation is hyperbolic. Every non-abelian group F of hyperbolic transformations is properly discontinuous except in points of the real axis, and so the totality of such groups F is identical with the totality of fundamental groups of closed or bounded bilateral surfaces φ of finite or infinite connectivity. Thus the abstract characterization of the groups F is well known. For closed φ the classes of automorphisms of F correspond to the classes of continuous representations of φ of degree ±1; especially classes of degree $+1$ are taken into consideration, and their group Γ_p, for genus p of φ, is investigated. Γ_p can be generated by a finite number of classes of representations, represented by so-called screw-transformations (Schraubungen), the type of which can be described by replacing the generator a_1 by a_1b_1, not altering $b_1, a_2, \cdots, b_p$, if $\Pi_{i=1}^{p}(a_ib_ia_i^{-1}b_i^{-1})=1$ is the defining relation of F_p. The proof of this theorem and closer investigations of Γ_p have previously been given at length by the author [Acta Math. **69**, 135–206 (1938)]. If one does not restrict one's self to using screw-transformations as generators, Γ_2 can be generated by two elements and Γ_p, $p>2$, by three elements, one of which is of order $2(p+1)$, one of order p and one (a screw-transformation) of infinite order. Moreover it is found that the factor-group of the commutator-group of Γ_p is cyclic, its order being 12 for $p=1$, a divisor of 10 for $p>1$ and, in particular, 2 or 1 for $p>2$. Finally a brief review of the historical origin of the problem is given. *J. Nielsen* (Copenhagen).

Nielsen, Jakob. Über Gruppen linearer Transformationen. Mitt. Math. Ges. Hamburg **8**, part 2, 82–104 (1940). [MF 4116]

Let U be the unit circle $z\bar{z}=1$ in the complex z-plane and let D be its interior. Let F be a non-abelian group of linear fractional transformations with U as fixed circle, each transformation of F taking D into D, such that every transformation of F other than the identity is a hyperbolic transformation. The principal results of the paper are two theorems which are derived by elegant geometric methods. The first is that F is properly discontinuous in D. As stated, a different proof of this theorem has been given by Lauritzen [Mat. Tidsskr. B. **1939**, 69–76; these Rev. **1**, 257]. [It may be remarked that the isometric circle of Ford [cf. Ford, Automorphic Functions] is introduced and used to prove certain known results.] Secondly, let L be the set of limit points of F. The "convex figure" C_L of F is defined to be the product of all convex sets in $D+U$ (where the lines in terms of which convexity is defined are arcs of circles orthogonal to U) which contain L. The set C_L is compact mod F if there exists a positive number $r<1$ such that to any point of C_L there corresponds a congruent point at distance less than r from the origin. It is shown that compactness of C_L mod F is a necessary and sufficient condition that F have a finite set of generators. The author states that he will apply his methods to the problems of surface topology. *G. A. Hedlund* (Charlottesville, Va.).

Fubini, Guido. A remark on general Fuchsian groups. Proc. Nat. Acad. Sci. U. S. A. **26**, 695–700 (1940). [MF 3369]

Let A, B, C, D, X, Y be real matrices of order n, $|Y|\neq0$, $Z=X+iY$, $dZ=dX+idY$, $d\bar{Z}=dX-idY$, where the elements of dX and dY are independent differentials. It is proved that the trace ds^2 of the matrix $Y^{-1}(dZ)Y^{-1}(d\bar{Z})$ is invariant under the transformation $Z\to(AZ+B)(CZ+D)^{-1}$. If a group of such transformations maps onto itself a region R in which ds^2 is definite, and if it contains no infinitesimal transformations, then it is properly discontinuous in R. Hence the general Fuchsian groups introduced by Sugawara [Ann. of Math. (2) **41**, 488–494 (1940); these Rev. **2**, 37] are properly discontinuous in the space of the symmetric matrices Z with positive Y. *C. L. Siegel.*

Maass, Hans. Über Gruppen von hyperabelschen Transformationen. S.-B. Heidelberger Akad. Wiss. **1940**, no. 2, 26 pp. (1940). [MF 4122]

Let T be the space of n complex variables $\tau_k=x_k+iy_k$ with $y_k>0$ $(k=1, \cdots, n)$ and Γ a discontinuous group of simultaneous linear fractional transformations

$$\tau_k\to\frac{\alpha_k\tau_k+\beta_k}{\gamma_k\tau_k+\delta_k}$$

of T into itself. Since the differential form

$$ds^2=\sum_{k=1}^{n}y_k^{-2}(dx_k^2+dy_k^2)$$

is invariant under Γ, it defines a distance $\overline{PP^*}$ for any two points P and P^* of T. Let P_0 be any given point of T and $P_0, P_1, P_2, \cdots$ the system of images of P_0 under Γ. A fundamental domain F with respect to Γ is formed by the set of all points P of T satisfying the inequalities $\overline{PP_0}\leq\overline{PP_k}$ $(k=1, 2, \cdots)$. It is important for the theory of the corresponding automorphic functions to determine the shape of F in the neighborhood of the boundary of T. This is performed in the special case of Hilbert's modular group; then $\alpha_k, \beta_k, \gamma_k, \delta_k$ $(k=1, \cdots, n)$ run over the conjugates of all integers $\alpha, \beta, \gamma, \delta$ with $\alpha\delta-\beta\gamma=1$ in a totally real algebraic number field Z of degree n. In this case another construction of a fundamental domain F^* is given, using the properties of the ideals of Z. The domain F^* is bounded by a finite number of analytic manifolds, and there are exactly h inequivalent boundary points of F^* on the boundary of T, where h denotes the class number of Z. The latter result corrects an erroneous statement of Blumenthal, to whom the definition of F^* is due [Math. Ann. **56**, 509–548 (1903)]. *C. L. Siegel* (Princeton, N. J.).

Brenner, Joel. The decomposition theorem for abelian groups. Bull. Amer. Math. Soc. **47**, 116–117 (1941). [MF 3822]

It is proved by an inductive construction that an abelian group G has a basis if $p^kg=0$ for all $g\in G$, p prime, k fixed. *J. S. Frame* (Providence, R. I.).

Kontorovitch, P. Sur les groupes normalement décomposables. I. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 423–436 (1940). (Russian. French summary) [MF 3740]

A group G is said to be normally decomposable if each element belongs to a proper normal subgroup F_i of G. Such

a group can be written as a logical sum $G=F_1+F_2+\cdots+F_k$ of normal subgroups, whose members are said to form an invariant basis for G. If each F_i is a maximum invariant subgroup the basis is called a maximum invariant basis; if each F_i is also a maximum subgroup the basis is called a principal basis. The intersection $F_{123\cdots s}$ of s subgroups of the basis is called an intersection of order s; the intersection of all maximal invariant subgroups is called the common subgroup D; the intersection of the subgroups of a principal basis is called a principal subgroup F. The author proves among others the following theorems: (1) If a factor group of G is normally decomposable, so is G. (2) If the intersection $F_{12\cdots s}$, $s\leqq k$, of a maximum invariant basis is 1, but all intersections of lower order of $F_1, \cdots, F_s$ differ from 1, G is a direct product of simple groups having as center Z a direct product of abelian groups. (3) If G is normally decomposable with common subgroup $D\neq 1$, then G/D is normally decomposable with common subgroup 1. (4) A normally decomposable group G always has a principal basis with principal subgroup $F\neq 1$ (except when G is abelian of order p^2). (5) The factor group G/F is a direct product of elementary abelian groups. (6) The existence of an invariant subgroup F such that G/F is the direct product of elementary abelian groups is a necessary and sufficient condition that G be normally decomposable. (7) The author cites as examples of normally decomposable groups the so-called "special" groups, which are the direct products of their Sylow subgroups, and shows that a group is special when and only when it has an invariant basis composed of special groups. *J. S. Frame* (Providence, R. I.).

Schmidt, O. Über unendliche spezielle Gruppen. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 363–375 (1940). (Russian. German summary) [MF 3736]

"Special" groups G are groups G such that every proper subgroup H of G has a normalizer different from H. Any subgroup of a special group is special. The elements of finite order in a special group G form a normal subgroup M. The factor group G/M again is a special group; G/M does not contain any elements not equal to 1 of finite order. The group M is a direct product of p-groups (that is, of groups in which the order of every element is a power of a prime p). These results (and many others of a similar nature) have been obtained independently by R. Baer [Trans. Amer. Math. Soc. **47**, 393–434 (1940); these Rev. **2**, 1].

Schmidt considers infinite p-groups G in which the minimum condition for subgroups holds. This is the condition that G does not contain infinite sequences of subgroups $H_1\supset H_2\supset H_3\supset\cdots$ with $H_{i-1}\neq H_i$. As proved by Schmidt, the following four conditions are equivalent: (1) G is soluble. (2) Every finite set of elements generates a finite subgroup. (3) G is a special group. (4) G contains an abelian normal subgroup which is a direct product of a finite number of primary groups of type p^∞; and G/A is finite. The minimum condition for subgroups can be replaced by weaker conditions. An example of a p-group G is given which does not satisfy the minimum condition such that G is soluble, but not special; the center of G is the unit-element [cf. also the paper of Baer for such an example].

M. Marden and R. Brauer.

Coxeter, H. S. M. The binary polyhedral groups, and other generalizations of the quaternion group. Duke Math. J. **7**, 367–379 (1940). [MF 3408]

The relations $R^l=S^m=T^n=RST=Z$ (l, m, n integers) define an abstract group $<l, m, n>$ which is finite if $|l|^{-1}+|m|^{-1}+|n|^{-1}-1>0$. In such cases $g=2(|l|^{-1}+|m|^{-1}+|n|^{-1}-1)^{-1}$ is a positive integer. Excluding the trivial cases where either l, m or $n=0$ or 1, it is shown that the element Z generates an invariant subgroup of order $2u$ (later defined) whose quotient group is the polyhedral group of order g. For $l, m, n>1$ it is shown that the defining relations imply $Z^2=1$, and hence $u=1$. That they do not imply $Z=1$ is shown by giving an explicit representation of each of these cases, and of the corresponding binary polyhedral groups, by means of quaternions. When one or more of the exponents l, m, n are negative, but $|l|$, $|m|$, $|n|$ and $|l|^{-1}+|m|^{-1}+|n|^{-1}$ are each greater than 1, it is shown that $<l, m, n>$ has a subgroup $<|l|, |m|, |n|>$ except in the cases $<-2, 2, n>$, n odd, and $<-3, \pm 2, 3>$. But in all these finite cases it is shown that $Z^{2u}=1$, where $u=(1-l^{-1}-m^{-1}-n^{-1})(|l|^{-1}+|m|^{-1}+|n|^{-1}-1)^{-1}$. The special cases are studied in detail, and the paper concludes with a remark about finite factor groups of $<2, 3, 6>$, $<2, 4, 4>$ and $<3, 3, 3>$. *J. S. Frame* (Providence, R. I.).

Morosoff, V. V. On primitive groups in three variables. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 193–212. (Russian) [MF 3513]

Classification of primitive groups in three variables by algebraic means. If $Y=a_{\alpha\beta}x^\alpha p_\beta$, $\alpha, \beta=1, 2, 3$, is any operator of the primitive group, the matrices $\|a_{\alpha\beta}\|$ form a Lie ring whose basis may be chosen in two canonical forms depending on whether the characteristic roots are simple or not. The following 7 types of groups are obtained: the projective group G_{15}, the general affine group G_{12}, a unimodular group G_{11}, a G_{10} which is similar to the projective group of the general linear complex, a G_7, a G_6 similar to the group of motions in Euclidean space and another G_6 which is similar to the group of motions in non-Euclidean space. *M. S. Knebelman* (Pullman, Wash.).

Magnus, Wilhelm. Über Gruppen und zugeordnete Liesche Ringe. J. Reine Angew. Math. **182**, 142–149 (1940). [MF 3427]

The paper contains a report on results concerning relations between groups and certain associative rings and Lie rings. Some of these results have been given in previous papers of the author [Math. Ann. **111**, 259–280 (1935); J. Reine Angew. Math. **177**, 105–115 (1937); Monatsh. Math. Phys. **47**, 307–313 (1939)], while others are new. Let $\mathfrak{G}$ be a group with a finite number of generators so that $\mathfrak{G}$ appears as a factor group $\mathfrak{F}/\mathfrak{N}$ of a free group $\mathfrak{F}$ with the generators $a, b, \cdots$. Set

$$a=1+x,\quad b=1+y,\quad a^{-1}=1-x+x^2-\cdots,$$
$$b^{-1}=1-y+y^2-\cdots,$$

where $x, y, \cdots$ are generating elements of a free associative, non-commutative ring $\mathfrak{R}$. Every element W of $\mathfrak{F}$ appears in the form $W=1+\sum d_i(x, y, \cdots)$, where in d_i the terms of degree i in $x, y, \cdots$ are collected. The elements W, for which $d_1=d_2=\cdots=d_{n-1}=0$, form a normal subgroup $\mathfrak{F}_n$ of $\mathfrak{F}$, and the series $\mathfrak{F}=\mathfrak{F}_1\supset\mathfrak{F}_2\supset\cdots$ is identical with the lower central series of $\mathfrak{F}$. Let Λ be a free Lie ring with the generators $\xi, \eta, \cdots$, and associate every element of Λ with an element of $\mathfrak{R}$ such that (1) $\xi\to x$, $\eta\to y$, $\cdots$. (2) If $\varphi\to f$ and $\psi\to g$, then $\varphi\pm\psi\to f\pm g$ and $[\varphi\psi]\to fg-gf$. The images of the homogeneous polynomials $\delta_n(\xi, \eta, \cdots)$ of degree n with integral rational coefficients form a module $\mathfrak{P}_n$ in $\mathfrak{R}$. It is shown that $\mathfrak{P}_n$ consists of those polynomials d_n of $x, y, \cdots$ which appear as second term in the expansion $W=1+d_n+\cdots$

of elements W of $\mathfrak{F}_n$. If W is restricted to the subgroup $(\mathfrak{N} \cap \mathfrak{F}_n) \cdot \mathfrak{F}_{n+1}$ of $\mathfrak{F}_n$, the corresponding d_n form a module $\mathfrak{M}_n$. If $\mathfrak{P}_n$ and $\mathfrak{M}_n$ are taken as additive groups, then $\mathfrak{P}_n/\mathfrak{M}_n \cong \mathfrak{A}_n$, where $\mathfrak{A}_n$ is the nth factor group of the lower central series $\mathfrak{G}=\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \cdots$ of $\mathfrak{G}$, that is, $\mathfrak{P}_n/\mathfrak{M}_n \cong \mathfrak{G}_n/\mathfrak{G}_{n+1} \cong \mathfrak{F}_n/(\mathfrak{N} \cap \mathfrak{F}_n) \cdot \mathfrak{F}_{n+1}$. The elements δ_n of Λ which are mapped upon elements of $\mathfrak{M}_n$ form a module $\mathfrak{M}_n$ in Λ. It is shown that the sum of all $\mathfrak{M}_n$ is an ideal $\mathfrak{M}$ of Λ. The Lie quotient ring $\Lambda^*=\Lambda/\mathfrak{M}$ is denoted as the Lie ring associated with the group $\mathfrak{G}$. It is uniquely determined by $\mathfrak{G}$, independent of the choice of the generators. If $\mathfrak{G}_k=\mathfrak{G}_{k+1}$, then $\Lambda^{*k}=0$, so that Λ^* is nilpotent in this case. If $\mathfrak{G}$ is a p-group, then $\mathfrak{G}$ and Λ^* contain the same number of elements. Conversely, assume that a Lie ring Λ^* is given which can be obtained from a free Lie ring Λ by prescribing a number of homogeneous relations for the generators $\xi, \eta, \cdots$. We may ask whether Λ^* appears as the associated ring of a suitable group $\mathfrak{G}$. It is proved that this is actually the case when $\Lambda^{*k}=0$ with $k \leqq p$ and $p^m\Lambda^*=0$ for some integer m. On the other hand, an example of a Lie ring with $\Lambda^{*p+2}=0$ is given, which does not belong to a group $\mathfrak{G}$. Finally, methods are discussed which give the construction of the invariants of degree m in the generators $\xi, \eta, \cdots$ of Λ with regard to the group of all non-singular linear transformations of $\xi, \eta, \cdots$.

R. Brauer (Toronto, Ont.).

Gruschko, I. Über die Basen eines freien Produktes von Gruppen. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 169–182 (1940). (Russian. German summary) [MF 3197]

Let $\mathfrak{G}=\mathfrak{AB}\mathfrak{C}\cdots$ be the free product of discrete groups $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \cdots$, each factor group having a finite set of generators. Every element G of $\mathfrak{G}$ has a unique representation $M_1M_2\cdots M_k$, such that each factor M_i belongs to one of the factor groups, $M_i \neq 1$, and two consecutive factors M_i, M_{i+1} belong to two distinct factor groups. The integer k is the length of G: $k=l(G)$. If $g_1, \cdots, g_m$ is a set of generators of $\mathfrak{G}$, then $\sum l(g_i)$ is the length of the base $g_1, \cdots, g_m$. If $\mathfrak{g}_1, \cdots, \mathfrak{g}_m$ denote free generators of a free group, then to any other set of generators $\mathfrak{g}_1', \cdots, \mathfrak{g}_m'$ of this free group there corresponds in an obvious fashion a base $g_1', \cdots, g_m'$ of $\mathfrak{G}$. Any such set of generators is called an admissible base with respect to the base $g_1, \cdots, g_m$. A minimal base is an admissible base of least length. The following theorem is proved: if $g_1, \cdots, g_m$ is a minimal base, then each g_i lies in one of the factor groups. The elements g_i which lie in a given factor group, say $\mathfrak{A}$, form a minimal base for A.

O. Zariski (Cambridge, Mass.).

Fouxe-Rabinovitch, D. I. Über die Automorphismengruppen der freien Produkte. I. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 265–276 (1940). (Russian. German summary) [MF 3463]

Given the free product A of a finite number of irreducible groups A_i $(i=1, 2, \cdots, n)$ and the automorphism groups Φ_i^* of A_i, the author constructs the automorphism group Γ of A. There is a restrictive hypothesis, which the author promises to remove in a succeeding paper, that none of the subgroups A_i is an infinite cyclic group. The group Γ is shown to be generated by subgroups Γ_{ij}, $i \neq j$, isomorphic to A_i, and groups Φ_i isomorphic to Φ_i^*; and to these must be added a group Ω isomorphic to a direct product of a certain number of symmetric groups of appropriate orders, to take account of isomorphisms (if they exist) between groups A_i and A_j, $i \neq j$. Earlier work here by Golowin and Szadowsky [Rec. Math. [Mat. Sbornik] N.S. **4** (**46**), 505–514 (1938)] is confined to the cases $n=2, 3$. The striking feature of the paper is the determination of the complete set of defining relations of Γ. This had been accomplished previously for the case $n=2$ only.

L. Zippin.

Fuchs-Rabinowitsch, D. J. On a certain representation of a free group. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 154–157 (1940). (Russian) [MF 3310]

Let A, B be two-rowed square matrices over a commutative field K [of characteristic zero, we believe, although this assumption is not stated explicitly]. It is proved that if A and B satisfy certain conditions then the multiplicative group generated by them is a free group. The conditions are as follows: (1) $|A|=|B|=1$; (2) Trace A = Trace $B=a$; (3) $A^n \neq 1$, $B^n \neq 1$, if $n \neq 0$ is an integer; (4) if $|A-B|=d$, then d is transcendental with respect to the field $R(a)$, where R is the field of rational numbers; (5) AB, A, B, 1 are linearly independent over K. From this result follows, by a well-known theorem, that also the matrices A^mBA^{-m} $(m=0, 1, 2, \cdots)$ are free generators of the group generated by them. The special matrices

$$A=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, \quad B=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$$

satisfy the above conditions 1–5, provided x is a transcendental quantity. Hence the matrices

$$X_m=\begin{pmatrix}1-mx, & x\\ -m^2x, & 1+mx\end{pmatrix}, \quad m=0, 1, 2, \cdots,$$

furnish a representation of a free group with a denumerable set of free generators. By means of this representation a new proof is given of a theorem of Magnus to the effect that the identity is the only element which is common to all the commutator groups of a free group. This last theorem is extended to free products of abelian groups.

O. Zariski (Cambridge, Mass.).

Doniakhi, Kh. A. Linear representation of the free product of cyclic groups. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 158–165 (1940). (Russian) [MF 3311]

The main result obtained in this paper is the proof of the existence of a representation by two-rowed matrices of the free product of cyclic (finite or infinite) groups. This representation is irreducible. In the special case of free groups this result was proved by Fuchs-Rabinowitsch [see the preceding review], and the present paper follows a similar method. The generating matrices are constructed explicitly. In the construction of a free group generated by two two-rowed matrices A and B, given by Fuchs-Rabinowitsch, it was assumed that 1, A, B, AB are linearly independent. The author observes that this condition is necessary. Namely, he makes the following interesting remark: if 1, A, B, AB are linearly dependent, then the matrices A and B satisfy the following relation:

$$A\,B\,A^{-1}\,B\,A\,B^{-2}\,A^{-1}\,B\,A\,B\,A^{-1}\,A^{-2}=1.$$

O. Zariski (Cambridge, Mass.).

Brauer, Richard. On the Cartan invariants of groups of finite order. Ann. of Math. (2) **42**, 53–61 (1941). [MF 3673]

If G is a group of finite order g, p a prime number, then let $g/n_1, \cdots, g/n_k$ be the numbers of elements in the differ-

ent classes $C_1, \cdots, C_k$ of conjugate elements of orders prime to p. Let P be the determinant of the matrix $[p_i(C_j)]$, where the p_i are the absolutely irreducible characters modulo p of G (with values in the field of complex numbers!). Let C be the determinant of Cartan's p-matrix whose coefficients are the multiplicities with which the absolutely irreducible modular representations of G occur in the indecomposable parts of the regular modular representation of G. R. Brauer and C. Nesbitt [University of Toronto Studies, Math. Series no. 4, 1937] have shown that $P^2C=\pm n_1 \cdots n_k$ and that P is prime to p. The author proves that C is the highest power of p dividing $n_1 \cdots n_k$ in proving that P^2 is the greatest factor of $n_1 \cdots n_k$ which is prime to p.

R. Baer (Urbana, Ill.).

Wigner, Eugene P. On representations of certain finite groups. Amer. J. Math. **63**, 57–63 (1941). [MF 3629]

If $\zeta(R)$ represents the number of elements S in a finite group G such that $S^2=R$ [cf. Frobenius and Schur, S.-B. Preuss. Akad. Wiss. 1906, 186–208], and v_R is the number of elements of G permutable with R, then the author proves that

$$\sum_R (\zeta(R))^2 \leqq \sum_R (v_R)^2.$$

Equality holds only for those groups G such that (a) every element of G is equivalent to its inverse, and (b) the direct product of any two irreducible representations of G contains no irreducible representation more than once. It is noted that for generality the character $[j; C]$ in formula (3a) should be replaced by its conjugate $[j; C]^*$, and similar changes made in (11) and (14a); but the results are not affected. This interesting theorem yields information concerning the symmetric and antisymmetric parts of the direct square of any irreducible representation of a group satisfying (a) and (b). *G. de B. Robinson* (Toronto, Ont.).

Lee, H. C. On the representations of the complex 3-dimensional and the real 4-dimensional orthogonal groups and their isomorphism. J. Chinese Math. Soc. **2**, 225–233 (1940). [MF 4209]

"It is well known that the complex linear unimodular group G_3 of a 2-dimensional affine space E_2 is a representation of the real orthogonal group G_4 of a 4-dimensional Euclidean space R_4. In this paper it is shown that the same group G_3 is also a representation of the complex orthogonal group G_3 of a 3-dimensional Euclidean space R_3, and that an isomorphism between G_4 and G_3 can be established." [Author's introduction.] *G. de B. Robinson.*

Thomas, L. H. On unitary representations of the group of De Sitter space. Ann. of Math. (2) **42**, 113–126 (1941). [MF 3677]

The group considered is the continuous group of transformations which leave the quadratic form $w^2+x^2+y^2+z^2-t^2$ invariant. Its unitary differentiable one or two valued representations are determined by a consideration involving the ten infinitesimal elements of the group. Six of these infinitesimal elements span a subgroup which leaves both t and $w^2+x^2+y^2+z^2$ invariant and is therefore isomorphic with the group of rotations in four dimensional Euclidean space. The (well-known) unitary irreducible representations of this group can be characterized by two numbers j_1 and j_2, both either integer or half integer. The whole representation is assumed to be in a form in which the matrices corresponding to the above subgroup are in the reduced form. The four remaining infinitesimal elements represent transformations in the wt, xt, yt, zt planes, respectively. The commutation relations show that they have matrix elements at the intersection of such rows and columns the j_1 of which differ by $\pm\frac{1}{2}$. The same holds for the j_2 of these rows and columns. Thus the sum j_1+j_2 is either an integer or a half integer for all those representations of the subgroup which occur in an irreducible representation of the whole group. The irreducible representations of the whole group can be characterized by two invariants which correspond, in the terminology of physics, to mass and spin. For every representation of the whole group, the representations of the subgroup which are contained therein are enumerated. The j_1 and j_2 run from their respective minimum values to infinity, assuming all values compatible with the integer or half integer character of j_1+j_2. The minimum values of j_1 and j_2 depend on the representation. All representations have infinitely many dimensions. *E. P. Wigner.*

Malcev, A. On isomorphic matrix representations of infinite groups. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 405–422 (1940). (Russian. English summary) [MF 3739]

This is concerned, in a fundamental way, with the class of groups which admit an isomorphic representation by matrices of some finite degree with elements from some field. This class we shall here denote by R; by R_c, R^n and R_c^n, respectively, we denote the subclasses for which the characteristic of the field has the preassigned value c, the degree of the matrices is n, and both are preassigned. For the most part, the theorems of the paper are not concerned with any particular field.

Necessary and sufficient conditions are given, for every n and c, that an abelian group belong to R_c^n. A sufficient condition that a periodic group belong to R_c, for whatever c, is that it have an invariant abelian subgroup of finite index which belongs to R_c; in the particular case that c is zero, this condition is also necessary. If a p-group is in R, every finite set of its elements generates a finite group. An infinite p-group is in R_0 if and only if it is special in the sense of Tschernikow [Rec. Math. [Mat. Sbornik] **6** (**48**), 199–214 (1939); these Rev. **1**, 162].

An interesting study of the solvability in some field of a transfinite system of equations and inequalities whose finite subsystems are supposed solvable in some field leads to the theorem: if every finite set of the elements of a group G generates a group in R^n, then G is in R^n. This leads to a consideration of the representation of groups with a finite number of generators. A theorem here is to the effect that if such a group belongs to R_0^n it is a "limit," in a quite reasonable sense, of a sequence of finite groups which belong to various R_p^n, for primes p. Another theorem states that if such a group belongs to R_p^n it is a "limit" of a sequence of groups isomorphically representable upon finite fields of the same characteristic, and with the same degree. An application of the results of this section yields: every representation, of degree greater than one, of an arbitrary p-group is reducible. The paper concludes with this contribution to the problem of Hopf: if a group with a finite number of generators belongs to R, then it is not isomorphic with any of its proper factor groups. *L. Zippin* (Flushing, N. Y.).

Wielandt, Helmut. p-Sylowgruppen und p-Faktorgruppen. J. Reine Angew. Math. **182**, 180–193 (1940). [MF 3432]

The method of monomial representations is used to prove the existence under certain circumstances of an invariant subgroup $\bar{\mathfrak{G}}$ of a given group $\mathfrak{G}$. The Sylow-group $\mathfrak{P}$ of $\mathfrak{G}$

is assumed to have a subgroup $\mathfrak{P}_1$, whose normalizer $\mathfrak{N}$ in $\mathfrak{G}$ has an invariant subgroup $\overline{\mathfrak{N}}$ such that $\mathfrak{N}/\overline{\mathfrak{N}}$ is an Abelian p-group. To prepare the reader for the assumptions of the main theorem the author states three definitions. (1) An exceptional group (Ausnahmegruppe) is the name applied to a transitive group $\mathfrak{T}$ of monomial substitutions on p variables if its order is a power of p and if it contains a diagonal substitution D whose multipliers ϵ_r are pth roots of unity with product $\Delta(D)\neq 1$. It is shown that each such group $\mathfrak{T}$ is irregular as defined by P. Hall. (2) A p-group $\mathfrak{P}$ is said to have the property R if no exceptional group appears among the monomial substitution groups which represent $\mathfrak{P}$ or its subgroups. Every regular p-group has the property R. (3) If $\mathfrak{A}$ is a subgroup of $\mathfrak{H}$, and $\mathfrak{H}$ a subgroup of $\mathfrak{G}$, then $\mathfrak{A}$ is called strongly closed in $\mathfrak{H}$ with respect to $\mathfrak{G}$ if the only elements of $\mathfrak{H}$ conjugate to $\mathfrak{A}$ in $\mathfrak{G}$ are those of $\mathfrak{A}$; weakly closed if the only subgroup of $\mathfrak{H}$ conjugate to $\mathfrak{A}$ in $\mathfrak{G}$ is $\mathfrak{A}$ itself. The strong closure $(\mathfrak{A})^*$ of an arbitrary subgroup $\mathfrak{A}$ of $\mathfrak{H}$ is the union of all $G^{-1}\mathfrak{A}G\wedge\mathfrak{H}$. Several lemmas lead to the main theorem. Th. 1. Let the subgroup $\mathfrak{P}_1$ be weakly closed in the Sylow group $\mathfrak{P}$ of $\mathfrak{G}$. Let the normalizer $\mathfrak{N}$ of $\mathfrak{P}_1$ have an invariant subgroup $\overline{\mathfrak{N}}$ with Abelian p-factor group $\mathfrak{N}/\overline{\mathfrak{N}}$. Let $\mathfrak{P}$ have subgroups $\mathfrak{P}_0$ and $\mathfrak{P}_2$, such that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are invariant in $\mathfrak{P}_0$, $\mathfrak{P}_0$ is strongly closed in $\mathfrak{P}$, the strong closure of $\mathfrak{P}_2$ lies in $\overline{\mathfrak{N}}$, and for each pair of elements $P_0\subset\mathfrak{P}_0$, $P_1\subset\mathfrak{P}_1$ the factor group $\{P_0, P_1, \mathfrak{P}_2\}/\mathfrak{P}_2$ has the property R. Then $\mathfrak{G}$ has an invariant subgroup $\overline{\mathfrak{G}}$ such that $\overline{\mathfrak{G}}\wedge\mathfrak{N}=\overline{\mathfrak{N}}$, $\overline{\mathfrak{G}}\mathfrak{N}=\mathfrak{G}$, and hence $\mathfrak{G}/\overline{\mathfrak{G}}\backsimeq\mathfrak{N}/\overline{\mathfrak{N}}$. In the thirteen stages of the proof, a monomial representation of $\mathfrak{G}$ is set up wherein $\mathfrak{N}$ is the subgroup of index n which leaves x_1 fixed and $\overline{\mathfrak{N}}$ is the subgroup which gives x_1 the factor 1. It is shown that $\mathfrak{P}_1$ has but one transitive constituent of degree 1, and that all the diagonal factors of an element P_0 are equal. Finally it is shown that the subgroup $\overline{\mathfrak{G}}$ of $\mathfrak{G}$ for which the factor product $\Delta(\overline{\mathfrak{G}})$ is 1 forms the required invariant subgroup. The condition "$\mathfrak{N}/\overline{\mathfrak{N}}$ abelian" is replaced in Th. 8 by "$\mathfrak{P}_1\mathfrak{N}/\overline{\mathfrak{N}}$ abelian," and in Th. 10 by "$\mathfrak{P}_1$ strongly closed in $\mathfrak{P}$." Numerous special cases of the main theorem are stated as illustrations. For example, if the order of $\mathfrak{G}$ is not divisible by p^{p+1}, then the maximal p-factor group of $\mathfrak{G}$ is isomorphic to that of the normalizer of a Sylow p-group of $\mathfrak{G}$. An example is given to show that without all the hypotheses of the main theorem the existence of the Abelian p-factor group $\mathfrak{N}/\overline{\mathfrak{N}}$ need not imply the existence of an invariant subgroup $\overline{\mathfrak{G}}$. *J. S. Frame.*

Tschernikow, S. Über Gruppen mit einer Sylowschen Menge. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 377–394 (1940). (Russian. German summary) [MF 3737]

This continues the author's investigations of infinite periodic groups [Rec. Math. [Mat. Sbornik] N.S. **6** (**48**), 199–214 (1939) and **7** (**49**), 35–64 (1940); cf. these Rev. **1**, 162 and **2**, 5] from the point of view of a new set of concepts. The principal new term and its definition are as follows. A Sylow-set of a periodic group G is an ordered set M of normal subgroups of G (containing G and also the identity) such that (1) if $A\subset B$ are elements of M and B/A is no p-group, then there is an element of M between A and B; (2) if $A\subset B\subset C$ are elements of M, then C/B and B/A have no elements of the same order (the identity excepted); (3) the sum and intersection of an arbitrary set of elements of M is contained in M. Another concept of some interest is the following: the element b of a periodic G is said to be Sylow-separable from an element c if there exists some normal group B which contains b, but not c, and is such that the sets B and $G-B$ have no elements whose orders are powers of the same prime. *L. Zippin* (Flushing, N. Y.).

Gelfand, I. and Raikov, D. On the theory of characters of commutative topological groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 195–198 (1940). [MF 3599]

This paper deals with the characters of a commutative topological group S, in which an invariant measure, with the usual properties of regular Lebesgue measure, is given. (It is convenient to assume that every single point in S has measure 0.) An elegant, quasi-algebraical method is used: The space L of all complex numerical functions $x(g)$, $g\in S$, with finite $\|x\|=\int x(g)dg$ is introduced. L is a normed ring, with $\|x\|$ as norm, and the "convolution" $x*y\,(g)=\int x(g-h)y(h)dh$ as multiplication; L possesses no unit, but it may be extended to a normed ring R with unit e by symbolic adjunction of e. The general element of R is then $\lambda e+x$, λ a complex number, $x\in S$, with norm $|\lambda|+\|x\|$.

L is a maximal ideal in R. The main device of this investigation consists in showing that the other maximum ideals M in R are in a one-to-one correspondence with the (continuous) characters $\chi(g)$ of S. This correspondence is defined by

$$\text{Restclass of } \lambda e+x \text{ in } R/M=\lambda+\int x(g)\overline{\chi(g)}dg,$$

or equivalently by

$$\chi(h)=\frac{\text{Restclass of } x_h \text{ in } R/M}{\text{Restclass of } x \text{ in } R/M}.$$

(Here $x_h(g)\equiv x(g+h)$, and the above quotient is really independent of x.) This procedure permits quick derivations of several important facts; e.g.: (1) All measurable characters of S are also continuous. (2) $\lambda e+x$ possesses a reciprocal (in R) if and only if $\lambda\neq 0$ and $\lambda+\int x(g)\overline{\chi(g)}dg\neq 0$ for all characters $\chi(g)$ of S. (3) R is semisimple. (4) If $g_1\neq g_2$, then a character with $\chi(g_1)\neq\chi(g_2)$ exists. *J. von Neumann.*

Drinfeld, G. Sur les opérateurs permutant les invariants intégraux d'un groupe continu de transformations. II. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 5, 117–122 (1940). (Ukrainian. Russian and French summaries) [MF 3753]

G_r is an r-parameter Lie group and I is any integral invariant of G_r. If Z is the symbol of a one parameter Lie group such that ZI is also an integral invariant of G_r and if $X_1, \cdots, X_r$ are the symbols of G_r, it is shown that G_r must be an invariant subgroup of G_{r+1} generated by the $r+1$ symbols $Z, X_1, \cdots, X_r$. The method of proof is the same as in the first part for the case $r=1$ [cf. the same Rec. Trav. **1940**, no. 4, 157–164 (1940); these Rev. **2**, 127]. *M. S. Knebelman* (Pullman, Wash.).

Sushkevich, A. K. Investigations on infinite substitutions. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 245–253. (Russian) [MF 3516]

A one to one mapping $\binom{n}{\varphi(n)}$ of the set $N\equiv[n]$ of natural numbers into itself is called by the author an ordinary infinite substitution and such a mapping of N into its proper part a deficient substitution. A many to one mapping of N into itself is called a redundant substitution and of N into its proper part a mixed substitution. The author speaks of a set with an associative binary operation as a generalized group. A left group is a generalized group with the left

division (that is, in it $XA=B$ has a solution X for every pair of elements A, B) and a left semigroup is a g.g. with the right cancellation law (that is, $X_1A=X_2A$ implies $X_1=X_2$ [cf. Sushkevich, Math. Ann. **99**, 30–50 (1928)]). The following are the chief results of the present paper: Ordinary infinite substitutions form a group. [Regarding such groups, cf. Haar, Math. Z. **33**, 129–159 (1931); Schreier and Ulam, C. R. Acad. Sci. Paris **197**, 737–738 (1933); Studia Math. **4**, 134–141 (1933).] All the deficient infinite substitutions form a left semigroup, all the redundant ones a right semigroup. If $\mathfrak{G}$ is a group or a semigroup of ordinary substitutions and $E=KL$ a decomposition of the identity substitution E into the product of a deficient substitution K and a redundant substitution L, then $L\cdot\mathfrak{G}\cdot K$ is also either a group or a semigroup of mixed substitutions and is isomorphic with $\mathfrak{G}$. Conversely every group of mixed substitutions may be obtained in this manner. Employing decompositions $E=KL_i$ with different redundant right factors L_i, the author obtains the set $\mathfrak{Z}=L_1\mathfrak{G}K+L_2\mathfrak{G}K+\cdots$ which is a left group or semigroup. Conversely, every left group of infinite substitutions may be written in this form. Similarly one may write

$$\mathfrak{R}=\sum_{\mu=1}^{r}\sum_{\lambda=1}^{s}L_\lambda\mathfrak{G}K_\mu=\sum_{\lambda=1}^{s}\mathfrak{B}_\lambda=\sum_{\mu=1}^{r}\mathfrak{A}_\mu,$$

where $E=K_1L_\lambda=K_\mu L_1$ and $K_\mu L_\lambda$ are ordinary substitutions. $\mathfrak{R}$ is a "generalized group of kernel type" [Sushkevich, Commun. Soc. Math. Kharkow et Inst. Sci. Math. Ukraine **4** (1933) and **12** (1936)], and $\mathfrak{B}_\lambda=\sum_{\mu=1}^{r}L_\lambda\mathfrak{G}K_\mu$ and $\mathfrak{A}_\mu=\sum_{\lambda=1}^{s}L_\lambda\mathfrak{G}K_\mu$ are right and left groups, respectively, and $\mathfrak{G}_{\mu\lambda}=L_\lambda\mathfrak{G}K_\mu$ are semigroups or isomorphic groups.

A. E. Ross (St. Louis, Mo.).

Murdoch, D. C. Note on normality in quasi-groups. Bull. Amer. Math. Soc. **47**, 134–138 (1941). [MF 3828]

The author considers finite quasi-groups in which the product ab is always uniquely defined, in which the equations $ax=b$, $ya=b$ have unique solutions x and y, and for which there is a weak associative law. This law is equivalent to the assumption that G has a left co-set expansion relative to any subquasi-group H [Hausmann and Ore, Amer. J. Math. **59**, 983–1004 (1937)]. The author calls a subquasi-group H normal if $(aH)(bH)=(ab)H$ for a and b in G, and proves that those normal subgroups of G which contain all right units form a modular lattice. The analogous result found by Hausmann and Ore used a substantially different definition of normality, and required stronger substitutes for the associative law. The result holds for infinite quasi-groups if one strengthens the definition of normality to require $(ah)(bH)=(ab)H$, $(aH)(bh)=abH$. *S. MacLane.*

Murdoch, D. C. Structure of abelian quasi-groups. Trans. Amer. Math. Soc. **49**, 392–409 (1941). [MF 4339]

An abelian quasi-group is defined to be one in which the identity $(ab)(cd)=(ac)(bd)$ holds. The problem of constructing all abelian quasi-groups is solved. Every abelian quasi-group is the direct product of two abelian quasi-groups A and B, where A is such that every element is the solution of an equation $ax=a$ for some element a (depending on x) in A, and B contains an idempotent element. Both types can always be constructed from abelian groups by methods first used by Suschkewitsch [Trans. Amer. Math. Soc. **31**, 204–214 (1929)]. These results are achieved by using on quasi-groups the analogue of Schreier's extension of one group by another [Monatsh. Math. Phys. **34**, 165–180 (1926)]. *H. Campaigne* (Minneapolis, Minn.).

Brandt, H. Über die Axiome des Gruppoids. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 95–104 (1940). [MF 4408]

Several non-redundant sets of axioms for an earlier concept [Math. Ann. **96**, 360–366 (1926)] are shown to be equivalent. One set is the following: A gruppoid S is a set of elements A, B, C, $\cdots$, such that, for some pairs X, Y, a product XY is defined and is in S. (I) If $AB=A'B$, then $A=A'$. (II) If AB and BC exist, then $A(BC)$ and $(AB)C$ are defined and equal. (III) Given A and B, if there exists an X such that AX and BX are defined, then there exists a Y such that $YA=B$. (IV) Given A and B, there exists at least one element Z such that AZ and ZB are defined.

If the number of Z's in (IV) is finite, (III) may be discarded. *H. Campaigne* (Minneapolis, Minn.).

ANALYSIS

Thomas, Joseph Miller. Positive solutions of binomial inequalities. Duke Math. J. **7**, 291–297 (1940). [MF 3402]

Stimulated by certain systems of linear inequalities arising in the theory of partial differential equations the author investigates the consistency of such systems. The discussion is closely related to a paper by L. L. Dines [Ann. of Math. (2) **20**, 191–199 (1919)]. By a passage from the variables y_i to the new variables $x_i=e^{y_i}$ the linear inequalities reduce to inequalities of the form $m>p$, where m, p are monomials in the x_i's with positive exponents (not necessarily integral). For such systems a gradual reduction process is developed which always allows a decision as to the existence of positive solutions x_i. *I. J. Schoenberg* (Waterville, Me.).

Schmidt, Erhard. Über die Ungleichung, welche die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet. Math. Ann. **117**, 301–326 (1940). [MF 2999]

Let $z(t)$ be absolutely continuous in $0\leqq t\leqq\lambda$ and have at least one zero there. The author proves that, for any values of $a>0$ and $b\geqq 1$,

$$\left\{\frac{1}{\lambda}\int_0^\lambda |z|^a dt\right\}^{1/a}\leqq H\left(\frac{1}{a},\frac{b-1}{b}\right)\left\{\lambda^{b-1}\int_0^\lambda\left|\frac{dz}{dt}\right|^b dt\right\}^{1/b},$$

$$\frac{1}{\lambda}\int_0^\lambda \log|z|\,dt\leqq\log\left[\left(1/G\left(\frac{b-1}{b}\right)\right)\left\{\lambda^{b-1}\int_0^\lambda\left|\frac{dz}{dt}\right|^b dt\right\}^{1/b}\right],$$

where $G(u)=e^u u^{-u}\Gamma(1+u)$, $H(u,v)=G(u+v)/(G(u)G(v))$. It is proved that these inequalities remain true also in limiting cases $a=\infty$, $b=\infty$, and that they are the best possible. The functions for which the inequality sign must be replaced by the equality sign are completely determined. The proof is based on an elegant reduction of the problem to the consideration of the area of the curve

$$|\xi|^a+\eta^{b/(b-1)}=1,\qquad -1\leqq\xi\leqq 1;\ \eta\geqq 0.$$

J. D. Tamarkin (Providence, R. I.).

Iyengar, K. S. K. On Frullani integrals. Proc. Cambridge Philos. Soc. **37**, 9–13 (1941). [MF 3766]

The problem solved is that of giving a characterization of functions $\varphi(t)$, integrable over each interval $0<t_1\leqq t\leqq t_2<\infty$,

for which the Frullani integral

$$(1) \qquad I(a,b)=\lim_{t_1\to 0,\ t_2\to\infty}\int_{t_1}^{t_2}\frac{\varphi(at)-\varphi(bt)}{t}dt$$

exists for each pair of positive numbers a and b. Let $\rho=a/b$. Since the formula

$$(2) \qquad I(a,b)=\lim_{\epsilon\to 0}\int_{\epsilon\rho}^{\epsilon}\frac{\varphi(t)}{t}dt-\lim_{\epsilon\to 0}\int_{\epsilon\rho}^{\epsilon}\frac{\varphi(1/t)}{t}dt$$

holds whenever either member exists, the problem is reduced to that of existence of the first limit $I_1(\rho)$ in the right member of (2). It is shown that, if $\varphi(t)$ is integrable over each interval $0<t_1\leqq t\leqq t_2<\infty$, then $I_1(\rho)$ exists for each $\rho>0$ if and only if the two limits

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\varphi(t)dt, \quad \lim_{u\to 0}\frac{1}{u}\int_{0}^{u}\varphi(t)dt$$

both exist. *R. P. Agnew* (Ithaca, N. Y.).

Iyengar, K. S. K. On Frullani integrals. J. Indian Math. Soc. (N.S.) **4**, 145–150 (1940). [MF 4318]

Except for minor differences in exposition which favor the other, this paper is identical with the paper reviewed above. Neither of the papers gives reference to the other. *R. P. Agnew* (Ithaca, N. Y.).

Radó, Tibor. On a lemma of McShane. Ann. of Math. (2) **42**, 73–83 (1941). [MF 3675]

In an earlier paper [Ann. of Math. (2) **34**, 815–838 (1933)], McShane proved the following important lemma: In the unit square $Q: 0\leqq u\leqq 1,\ 0\leqq v\leqq 1$, let there be given triples of continuous functions $x(u,v)$, $y(u,v)$, $z(u,v)$, $x_n(u,v)$, $y_n(u,v)$, $z_n(u,v)$ such that the following conditions are satisfied: (I) The partial derivatives $x_u, x_v, y_u, y_v, z_u, z_v$ exist almost everywhere in Q. (II) The Jacobians $X=y_uz_v-y_vz_u$, $Y=z_ux_v-x_uz_v$, $Z=x_uy_v-x_vy_u$ are summable in Q. (III) The functions x, y and z are absolutely continuous on the boundary B of Q. (IV) We have:

$$\iint_Q X\,dudv=\tfrac{1}{2}\int_B(ydz-zdy), \quad \iint_Q Y\,dudv=\tfrac{1}{2}\int_B(zdx-xdz),$$

$$\iint_Q Z\,dudv=\tfrac{1}{2}\int_B(xdy-ydx).$$

(V) The functions $x_n(u,v)$, $y_n(u,v)$, $z_n(u,v)$ are quasi-linear in Q. (VI) $x_n\to x$, $y_n\to y$, $z_n\to z$ uniformly in Q. Then for each choice of the constants a, b and c there exists a sequence $\{V_n\}$ of measurable subsets of Q such that

$$\lim_{n\to\infty}\iint_{V_n}(aX_n+bY_n+cZ_n)dudv=\iint_Q(aX+bY+cZ)dudv.$$

The author gives an elegant proof of this lemma with the hypotheses (III) and (IV) omitted. *C. B. Morrey, Jr.*

Hestenes, M. R. Extension of the range of a differentiable function. Duke Math. J. **8**, 183–192 (1941). [MF 3954]

The author presents two methods for extending the range of a differentiable function. The first method, utilizing a reflection principle, is used to prove that, if $f(x_1,\cdots,x_n)$ is of class C^m, m finite, on a closed set A which possesses a suitably smooth boundary, then f may be extended to be of class C^m over the whole Euclidean n-space. The second method is similar to one used by Whitney [Trans. Amer. Math. Soc. **36**, 63–89 (1934)], and yields the result that a function of class C^m, m finite or infinite, on an arbitrary closed set A may be extended to be of class C^m over the whole space; moreover, in this case, the extension is of class C^m on the complement of A. *W. T. Reid.*

Popoviciu, Tiberiu. Notes sur les généralisations des fonctions convexes d'ordre supérieur (II). Acad. Roum. Bull. Sect. Sci. **22**, 473–477 (1940). [MF 3204]

[Notes III, V, VI, VII and VIII have appeared earlier; cf. these Rev. **1**, 204 and **2**, 76.] A real function $f(x)$ defined on a set E is said to be of order n if its divided differences formed over any $n+2$ distinct points of E have a constant sign. We say that (1) $E=E_1+E_2+\cdots+E_m$ is a decomposition of the linear set E in consecutive subsets if $x\epsilon E_i$, $y\epsilon E_j$, $i<j$, imply $x<y$. $f(x)$ is said to be piece-wise of order n in E if there is a finite consecutive decomposition (1) such that $f(x)$ is of order n in each of the E_i. The decomposition (1), connected as above with $f(x)$, is called a proper decomposition of E with respect to $f(x)$ if $f(x)$ is not of order n on any of the sets E_i+E_{i+1} $(i=1,\cdots,m-1)$. Now if $f(x)$ is piece-wise of order n in E, then the number m of terms of a corresponding decomposition (1) evidently has a minimum value h which is called the characteristic of $f(x)$. The chief result is the following theorem. The number m of terms of a proper decomposition (1) of E, with respect to $f(x)$, satisfies the double inequality $h\leqq m\leqq 2h-1$. The proof proceeds by induction for h. *I. J. Schoenberg.*

Boas, R. P., Jr. and Widder, D. V. Functions with positive differences. Duke Math. J. **7**, 496–503 (1940). [MF 3415]

Let $f(x)$ be real, defined for $a<x<b$. We make in turn the following two sets of assumptions concerning $f(x)$: (1) $f(x)$ is continuous and $\Delta_\delta^k f(x)\geqq 0$ provided $a<x<x+k\delta<b$; $k\geqq 2$ is fixed. (2) $f^{(k-2)}(x)$ exists in $a<x<b$ and is continuous and convex. The chief result is that (1) implies (2). The proof employs exclusively divided differences for equally spaced points. The authors point out that their result was previously established by T. Popoviciu [Mathematica, Cluj **8**, 1–85 (1934)] using, however, divided differences with unequally spaced points. Since furthermore (2) implies (1), the conditions (1) and (2) are seen to be equivalent. One aspect of the significance of results of this kind is brought out by the following remark. Denote by M_k the class of functions $f(x)$ subject to (2). Let $f_n(x)\epsilon M_k$ $(n=1,2,\cdots)$ and $f_n(x)\to f(x)$, where $f(x)$ is continuous; then $f(x)\epsilon M_k$. This closure property is immediately revealed by the equivalent definition (1) of the class M_k. *I. J. Schoenberg.*

Widder, D. V. Functions whose even derivatives have a prescribed sign. Proc. Nat. Acad. Sci. U. S. A. **26**, 657–659 (1940). [MF 3145]

The author proves the following remarkable result: let $f(x)$ have derivatives of all orders in $0\leqq x\leqq 1$, and let $(-1)^k f^{(2k)}(x)\geqq 0$, $k=0,1,2,\cdots$. Then $f(x)$ is an entire function. The proof is based on a consideration of the remainder of the Lidstone series associated with $f(x)$,

$$\sum_{n=0}^{\infty}[f^{(2n)}(1)\Lambda_n(x)+f^{(2n)}(0)\Lambda_n(1-x)],$$

combined with the fact that the polynomials $\Lambda_n(x)$ defined by the relations $\Lambda_0(x)=x$, $\Lambda_n''(x)=\Lambda_{n-1}(x)$, $\Lambda_n(0)=\Lambda_n(1)=0$, $n=1,2,\cdots$, have the sign of $(-1)^n$. *J. D. Tamarkin.*

Polya, George. On functions whose derivatives do not vanish in a given interval. Proc. Nat. Acad. Sci. U. S. A. **27**, 216–218 (1941). [MF 4196]

D. V. Widder has recently proved that if $f(x)$ satisfies

the conditions $(-1)^m f^{(2m)}(x) \geqq 0$ $(0 \leqq x \leqq 1;\ m=0, 1, \cdots)$ then $f(x)$ is an entire function of exponential type [cf. the preceding review]. A simpler proof of this result [to appear in the Bull. Amer. Math. Soc.] was subsequently given by R. P. Boas. The author now extends Widder's result in two different ways obtaining the following theorems. (I) Let α be a fixed real number, $0<\alpha<\pi$. If $f^{(n)}(x) \sin (n+1)\alpha \geqq 0$ for $0 \leqq x \leqq 1$ and $n=0, 1, \cdots$, then $f(x)$ is entire of exponential type. (II) If $f^{(4m)}(x) \geqq 0$ in $[0, 1]$ $(m=0, 1, \cdots)$ and $f^{(4m+1)}(x)$ reaches its maximum in $[0, 1]$ for $x=0$, then $f(x)$ is entire of exponential type. The proofs are based on an unexpected and subtle extension of Boas' proof mentioned above, also using Kronecker's theorem concerning the multiples $m\alpha$, mod 1, in the case of (I). A further joint note on this subject, by Boas and Polya, is to appear in the same journal. *I. J. Schoenberg* (Waterville, Me.).

Functional Analysis

Smiley, M. F. Measurability and modularity in the theory of lattices. Bull. Amer. Math. Soc. **47**, 76–78 (1941). [MF 3814]

The author continues his investigation into "measurability" with respect to a function μ on a lattice L [Bull. Amer. Math. Soc. **46**, 239–241 (1940); these Rev. **1**, 240]. He proves that, if all $x \leqq c$ are measurable, then they have two properties characteristic of modular lattices. The converse holds in the finite-dimensional case. *G. Birkhoff.*

Kolmogoroff, A. Kurven im Hilbertschen Raum, die gegenüber einer einparametrigen Gruppe von Bewegungen invariant sind. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 6–9 (1940). [MF 3538]

If a, x, y, u, v, Φ, ξ are elements of a Hilbert space H, if $y=Ux$ is a unitary transformation on H, if Kx has the form $a+Ux$ (a fixed, x variable) and if $K_t (-\infty < t < \infty)$ is a group for which $\|K_{t_n}x - K_t x\| \to 0$ as $t_n \to t$, then a function $\xi(t)$ from t to H can be represented in the form $\xi(t)=K_t x_0$ if and only if $\xi(t)$ can be put in the form

$$\lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} (e^{\Omega t}-1)\, d\Phi_\lambda + \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} (e^{\Omega t}-1)\, d\Phi_\lambda + ut + v,$$

where $(\Phi_{\lambda'}, \Phi_{\lambda''})=0$ for disjoint intervals λ', λ'' and the integral

$$\int_{-\infty}^{\infty} \min\,(1, \lambda^2)\, d\|\Phi_\lambda\|^2$$

exists in the Lebesgue-Stieltjes sense. This representation of $\xi(t)$ is unique, and it is closely related to another representation of the numerical function $B_\xi(\tau_1, \tau_2)=(\xi(t+\tau_1)-\xi(t), \xi(t+\tau_2)-\xi(t))$. The author might have quoted related work by Schoenberg [e.g., Bull. Amer. Math. Soc. **44**, 44 (1938)]. *S. Bochner* (Princeton, N. J.).

Kolmogoroff, A. N. Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 115–118 (1940). [MF 3542]

In the terminology of the paper reviewed above, a function $\xi(t)$ is said to belong to class $\mathfrak{A}$ if, in addition to the requirements mentioned there, corresponding to each real k, there exists a transformation $A_k = A = a+qUx$, $q>0$, such that $\xi(kt)=A_k \xi(t)$. In finite dimensional Hilbert space all functions of $\mathfrak{A}$ have the trivial form $ut+v$, but not so in general Hilbert space. Here the author finds for $B_\xi(\tau_1, \tau_2)$ the value $c[|\tau_1|^\gamma + |\tau_2|^\gamma - |\tau_2-\tau_1|^\gamma]$ with $c \geqq 0$, $0 \leqq \gamma \leqq 2$, the case $ut+v$ corresponding to $\gamma=2$. An interesting case is $\gamma=1$, which is characterized by the relation

$$(\xi(t_1)-\xi(s_2),\ \xi(t_2)-\xi(s_2))=0$$

$(s_1<t_1<s_2<t_2)$. For random functions $\xi(t)$ the latter condition expresses absence of correlations, and the author states that a significant special case of such functions has been introduced by N. Wiener in 1923 in connection with Brownian movement. *S. Bochner* (Princeton, N. J.).

Fréchet, Maurice. Sur les espaces distanciés. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 265–267. [MF 3519]

L'auteur démontre que tous les espaces distanciés de puissances $\leqq p$ (p étant un nombre cardinal fini ou transfini) sont isométriques de sous-ensembles d'un même espace vectoriel distancié, complet et tendu D_p défini comme il suit. Soit P un ensemble de puissance p. Chaque élément X de D_p est une fonction numérique bornée (réelle, par exemple) $x(\xi)$ de l'élément ξ variable sur P. On représente par $X+Y$ et par aX, où a est un nombre (réel), l'élément défini par $x(\xi)+y(\xi)$ et par $ax(\xi)$, respectivement. On appelle norme de X la borne supérieure $||X||$ de $|x(\xi)|$ quand ξ varie sur P. Remarquons que $D_{\aleph_0}$ a été étudié sous le nom D_ω ou (m). C'est une précision d'un résultat de Kunugui, d'après lequel tout espace distancié est isométrique d'un ensemble appartenant à un espace vectoriel, distancié et complet (c'est-à-dire, du type (B) de S. Banach), qui entraîne immédiatement un théorème de F. Hausdorff: tout espace distancié peut être prolongé à un espace distancié complet, en conservant la métrique. *K. Kunugui.*

Sobczyk, Andrew. Projections in Minkowski and Banach spaces. Duke Math. J. **8**, 78–106 (1941). [MF 3946]

This paper extends in certain directions the result of the reviewer that there exist manifolds in L_p and l_p $(p \neq 2)$ which do not possess projections [cf. Trans. Amer. Math. Soc. **41**, 138–152 (1937)]. A simpler and more symmetric treatment is given by considering the bounds of transformations in the form $2E-1$, rather than the bounds of projections. Sequence spaces or spaces with bases of symmetry are determined by an expanding sequence of manifolds with finite, multiplicatively increasing dimensionality and within each such manifold an Euclidean norm is introduced and compared with the spatial norm. The author characterizes a class of such spaces in which the existence of a projection on every manifold implies that the space is essentially Euclidean. The growth rate of certain functions of the dimensionality of these manifolds is also determined. *F. J. Murray* (New York, N. Y.).

Phillips, R. S. A characterization of Euclidean spaces. Bull. Amer. Math. Soc. **46**, 930–933 (1940). [MF 3450]

A proof of the theorem first given by Kakutani [Jap. J. Math. **16**, 93–97 (1939); cf. these Rev. **1**, 146] that, if a Banach space has a projection of bound one on every two dimensional subspace, the Banach space is Euclidean. This proof is more direct. The reviewer believes that "$e(2r(x))$" in the 8th line from below on page 931 should be "$e(A_\beta r(x))$," where A_β is a constant depending on β and that "$C_{\gamma_n} C_\beta$" in the fourth line of page 932 should be "C_{γ_n}." *F. J. Murray* (New York, N. Y.).

Sirvint, G. Schwache Kompaktheit in den Banachschen Räumen. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 199–202 (1940). [MF 3600]

I. Gelfand [Rec. Math. [Mat. Sbornik] N.S. **4** (**46**), 235–284, 286 (1938)] proved that a subset M of a separable Banach space E is strongly compact if and only if every sequence of linear functionals defined on E which converges weakly to zero converges uniformly to zero on M. In this paper it is stated without proof that the same result holds for sequentially weakly compact sets if uniform convergence is replaced by quasi-uniform convergence. This result is then applied to the problems of weakly completely continuous (w.c.c.) operations, that is, operations which map the unit sphere into sequentially weakly compact sets. In order that a linear operation from a Banach space E to a separable Banach space E_1 be w.c.c., it is necessary and sufficient that the adjoint operation maps every sequence of elements of E_1^* (=the conjugate space of E_1) which converges weakly as a sequence of functionals into a sequence of elements of E^* which converges weakly as a sequence of elements. Lastly, a general form of w.c.c. linear operations $y=U(x)$ from an arbitrary Banach space to the space C of continuous functions (defined on $0\leqq t\leqq 1$) is given by $y=f_t(x)$, where $f_t \varepsilon E^*$ $(0\leqq t\leqq 1)$ is such that $X(f_t)$ is continuous in t for each $X \varepsilon E^{**}$. *S. Kakutani* (Princeton, N. J.).

Day, Mahlon M. Reflexive Banach spaces not isomorphic to uniformly convex spaces. Bull. Amer. Math. Soc. **47**, 313–317 (1941). [MF 4184]

Milman [C. R. (Doklady) Acad. Sci. URSS (N.S.) **20**, 243–246 (1938)] and Pettis [Duke Math. J. **5**, 249–253 (1939)] have shown independently that a Banach space which is isomorphic to a uniformly convex space must be reflexive; that is, for $\beta \varepsilon B^{**}$ there is a $b \varepsilon B$ such that $\beta(f)=f(b)$ for every $f \varepsilon B^*$. The natural question of whether the converse is true is answered in the negative in the present note; in addition the author shows that reflexivity is not sufficient even in the presence of separability and strict convexity of the space. The author deduces from his results that certain general ergodic theorems of Alaoglu and G. Birkhoff [Proc. Nat. Acad. Sci. U. S. A. **25**, 628–630 (1939) and Ann. of Math. (2) **41**, 293–309 (1940); cf. these Rev. **1**, 148 and 339] have a wider range of applicability than the one which was originally announced by those authors. *J. A. Clarkson.*

LaSalle, J. P. Pseudo-normed linear spaces. Duke Math. J. **8**, 131–135 (1941). [MF 3948]

The author modifies Hyers' definition of a pseudo-norm [Duke Math. J. **5**, 628–634 (1939); cf. these Rev. **1**, 58] in such a way that the weakened triangle postulate of Hyers can be stated as follows: for any $e \varepsilon D$ there exists $f \varepsilon D$ such that $n(x+y, e)\leqq n(x, f)+n(y, f)$. The principal result is the following theorem. A necessary and sufficient condition that there exist no non-null linear functional on a pseudo-normed linear space T is that, for every $e \varepsilon D$, $U(e)_{\text{conv}}=T$, where $U(e)_{\text{conv}}$ is the convex hull of the set of $x \varepsilon T$ with $n(x, e)<1$. *J. V. Wehausen* (Columbia, Mo.).

Hildebrandt, T. H. On unconditional convergence in normed vector spaces. Bull. Amer. Math. Soc. **46**, 959–962 (1940). [MF 3456]

The notion of unconditional convergence of a series $\sum x_n$ of elements in a Banach space has been advanced in several equivalent forms. To two formulations originally proposed by Orlicz [Studia Math. **4**, 41–47 (1933)] have been added two others by Pettis. The author shows that these are in turn equivalent to a definition given by E. H. Moore (Moore's definition which applied to numbers may be instantly generalized to elements of a Banach space) which follows: Let σ be any finite subset $n_1, \cdots, n_k$ of the positive integers and denote $\sum_{i=1}^{k} x_{n_i}$ by $\sum_\sigma x_n$. Then $\sum_\sigma x_n$ is said to converge in the σ sense if the Moore-Smith limit: $\lim_\sigma \sum_\sigma x_n$ exists. It is also shown, following Moore, that if p is an index of an arbitrary set P and if $\lim_\sigma \sum_\sigma x(p)$ exists, where σ denotes a finite subset of P, then $x(p)=0$ except on a denumerable set $p_1, p_2, \cdots$ and $\sum x(p_n)$ converges unconditionally. *E. R. Lorch* (New York, N. Y.).

Maddaus, Ingo, Jr. On types of "weak" convergence in linear normed spaces. Ann. of Math. (2) **42**, 229–246 (1941). [MF 3681]

A linear set X of elements x is called an H-normed space if for each finite set of elements there is associated a non-negative number such that: (i) If $x_i=x_j$ then

$$\begin{aligned}\|(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n)\| &= \|(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_j, \cdots, x_n)\| \\ &= \|(x_1, \cdots, x_i, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)\|;\end{aligned}$$

(ii) $\|(x)\|=0$ implies $x=\bar{O}$, the null element of X; (iii) if a_i is real

$$\|(a_1x_1, \cdots, a_nx_n)\| \leqq \max_{1\leqq i\leqq n} |a_i| \, \|(x_1, \cdots, x_n)\|.$$

These axioms are three of the six for K-normed spaces studied by B. Vulich in papers cited. A sequence $\{x_n\}$ is H-convergent if given $e>0$ and a subsequence $\{x_{n_k}\}$ there exists a $K_0=K(e, \{x_{n_k}\})$ such that $k, s>K_0$ imply

$$\|(x_{n_1}-x_{n_s}, \cdots, x_{n_k}-x_{n_s})\|<e.$$

Also H-convergence to an element is defined. Theorems in H-convergence are proved and an example given to show that the H-limit of a sum is not necessarily the sum of the H-limits, even for two sequences.

An H-norm for the space of real numbers is defined by $\|(x_1, \cdots, x_n)\|=\inf(|x_1|, \cdots, |x_n|)$. Also a definition of an H-norm for each of the spaces C, L^p, c and l^p $(p\geqq 1)$ is given and in each case H-convergence of a sequence to an element is weaker than weak Banach convergence. Also in each case a different definition of an H-norm is given and the resulting H-convergence notion is shown to be equivalent to weak sequential convergence. Relations of H-convergence and K-convergence (of Vulich) are obtained by auxiliary convergence notions. Functions from an H-normed space to a Banach space, to an H-normed space and to other spaces are studied and implications of continuity in one sense from continuity in another obtained. The general form of the H-continuous linear functionals for each of the spaces C, L^p, c and l^p $(p\geqq 1)$ is obtained. *J. F. Randolph.*

Vulich, B. Une définition du produit dans les espaces semi-ordonnés linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 850–854 (1940). [MF 3578]

In any partially ordered linear space, satisfying axioms I to V of Kantorovitch [cf. Rec. Math. [Mat. Sbornik] **2** (**44**), 121–168 (1937)], and in which a Freudenthal unit exists, a product is defined for certain pairs of elements of the linear space by the following construction. Any positive element can be represented as the transfinite sum of positive multiples of elements of the Boolean algebra determined by the Freudenthal unit. The product of two positive elements is then obtained by the formal multiplication of two representations of the elements, where the product of two ele-

ments of the Boolean algebra is taken to be their inf. The product is independent of the representation used. The product of general elements is obtained by representing the elements as the difference of positive elements. It will exist only when the resulting transfinite sum makes sense. When it exists, the product is commutative, associative and distributive with respect to the addition. If the space is a space of functions over a set, then the definition of the product coincides with the notion of the product of the functions. *F. Bohnenblust* (Princeton, N. J.).

Vulich, B. Sur les propriétés du produit et de l'élément inverse dans les espaces semi-ordonnés linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 855–859 (1940). [MF 3579]

In the preceding paper the author introduced a product for elements of a partially ordered linear space. In the present paper the author discusses further properties of the product, in particular the existence and properties of inverse elements. Not every element has an inverse. In the particular case of function spaces, the inverse of a function is that function which assumes the reciprocal value of the given function when the latter is different from zero, and which is equal to zero otherwise. *F. Bohnenblust.*

Frink, Orrin, Jr. Series expansions in linear vector space. Amer. J. Math. **63**, 87–100 (1941). [MF 3632]

Given a subset A of a Banach space B, let A^C denote the closed linear manifold determined by A. A sequence $P=\{p_n\}$ of elements of B is called minimal if p_n is not contained in $(P-p_n)^C$ for any n. Given $x\varepsilon P^C$, the series (1) $\sum_{n=1}^{\infty} f_n(x)p_n$ is called its expansion, where $f_n(x)p_n$ is the projection of x on $(p_n)^C$ determined by this one-dimensional linear manifold and $(P-p_n)^C$. [The existence and properties of this projection might have been inferred from known results concerning these operators; see, for example, E. R. Lorch [Trans. Amer. Math. Soc. **45**, 217–234 (1939), Theorem 2.2].] A series $\sum_{\nu=1}^{\infty} x_\nu$ of elements of B is called semi-regular (s.r.) summable to x if the sequence of transforms $\sum_{\nu=1}^{\infty}\alpha_{\mu\nu}x_\nu$ converges to x, where the matrix $\|\alpha_{\mu\nu}\|$ is restricted only by the condition that $\alpha_{\mu\nu}\to 1$ as $\mu\to\infty$, for each ν. It is shown that (1) is the only series of p_n's which is s.r. summable to x, and that it is always so summable by a properly chosen (even triangular) matrix. If the functionals $\{f_n(x)\}$ are total, then (1) can not be s.r. summable to any other element. (The statement of this theorem in the paper's introduction might be misleading in that the extra condition is omitted.) Applications are made to Newton's interpolation series and power series. *J. A. Clarkson.*

Krein, Mark and Krein, Selim. On an inner characteristic of the set of all continuous functions defined on a bicompact Hausdorff space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 427–430 (1940). [MF 3219]

Let A be a (bi-)compact Hausdorff space and let $C(A)$ denote the set of all continuous functions φ defined over A. This $C(A)$ is a Banach space, $\|\varphi\|=\sup|\varphi(f)|$, $f\varepsilon A$, and a linear lattice, if $\varphi\geqq 0$ is defined as $\varphi(f)\geqq 0$ for $f\varepsilon A$. If 1 denotes the function $\varphi(f)=1$, then the norm $\|\varphi\|=$inf. of all real numbers t, such that $-t\cdot 1<\varphi<t\cdot 1$. The present authors sketch a proof of the converse statement: If E is a Banach lattice of elements x, in which a unit element 1 exists such that $\|x\|=\inf t$, $-t\cdot 1<x<t\cdot 1$, then E is isometric, lattice isomorphic with $C(A)$ for a certain compact A. The same results were announced simultaneously by S. Kakutani [Proc. Imp. Acad. Tokyo **16**, 63–67 (1940); cf. these Rev. **2**, 69]. This author has since obtained results enabling him to discuss the case when the existence of a unit element is not postulated. For other characterizations of spaces $C(A)$, in terms of its ring properties, see also M. H. Stone [Proc. Nat. Acad. Sci. U. S. A. **26**, 280–283 (1940); these Rev. **1**, 338] and I. Gelfand [C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 570–572 (1939); these Rev. **1**, 330]. *F. Bohnenblust* (Princeton, N. J.).

Šmulian, V. On multiplicative linear functionals in certain special normed rings. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 13–16 (1940). [MF 3540]

The author finds a representation for the general multiplicative linear functional ($f(xy)=f(x)f(y)$) over the normed rings of (1) bounded real functions over an arbitrary set Q, (2) over a bicompact space Q, (3) bounded continuous functions over a normal space Q. In each case the representation is given by $f(x)=\int_Q x(q)d\Phi(e)$, where $\Phi(e)$ is an additive function defined over a field of sets appropriate to the particular situation and assumes only the values 0 and 1. It is also shown, by a transfinite method, how such set-functions may be constructed. *J. V. Wehausen* (Columbia, Mo.).

Van der Lijn, Gaston. Les polynomes abstraits (Suite). Bull. Sci. Math. (2) **64**, 102–112 (1940). [MF 2573]

This continues the discussion of abstract polynomials [Bull. Sci. Math. (2) **64**, 55–80 (1940); these Rev. **1**, 259]. The abelian groups are specialized to linear spaces. The previous results are strengthened and new characterizations of polynomials appear. The lack of references is irksome, especially in view of the related work of Mazur and Orlicz [cf. Studia Math. **5**, 50–68, 179–189 (1935)]. *F. J. Murray* (New York, N. Y.).

Kantorovitch, L. A new method of solving of some classes of extremal problems. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 211–214 (1940). [MF 3604]

The paper is concerned with a weakly upper semi-continuous functional F defined on a convex and weakly compact subset A of a normed and linear space. If this functional F does not attain its maximum inside of A, it does so on the boundary; and there is a linear continuous functional which also attains its maximum at that point. The author's method is then essentially contained in a corollary of the result above. If f is an arbitrary linear continuous functional and A_f consists of all points of A in which f attains its maximum, then the maximum of F over A is precisely equal to the upper bound with respect to f of the number $p(f)$, the maximum of F on the set A_f. A criterion in terms of supporting hyperplanes of A is also given to enable one to know when the maximum is attained. Several illustrations of the author's method are indicated. *H. H. Goldstine.*

Randolph, John F. Metric separability and outer integrals. Bull. Amer. Math. Soc. **46**, 934–939 (1940). [MF 3451]

Let A be a plane set with finite outer linear (Carathéodory or Gillespie) measure L^*A. Let $f(p)$ be defined on A and bounded, and e_i the part of A for which $a_{i-1}\leqq f(p)<a_i$. If $\sum a_i L^*e_i$ tends to a limit as $a_i-a_{i-1}\to 0$, this limit is the outer integral $\int_A{}^* f(p)dp$ of f over A. Two plane sets A_1 and A_2 are metrically separated if $L^*(A_1+A_2)=L^*A_1+L^*A_2$; $f(p)$ defined on A is metrically separable relative to A if for every real λ the sets $f\geqq\lambda$, $f<\lambda$ are metrically separated. If $f(p)$ is bounded and metrically separable relative to each of two sets A_1, A_2, then $\int_{A_1+A_2}^* f dp$ exists. Let the plane set A be such that L^*A is finite, and let $\Gamma[A]$ be the points of the

complement of A at which the superior linear density of A is greater than zero. Then $B=A+\Gamma[A]$ is linearly measurable. If $f(p)$ is defined on A, bounded and metrically separable relative to A, then there exists $\varphi(p)$ measurable on B, $\varphi(p)=f(p)$ for p on A, and the Lebesgue integral of φ over B is equal to $\int_A{}^* f dp$. *R. L. Jeffery.*

Calkin, J. W. Abstract definite boundary value problems. Proc. Nat. Acad. Sci. U. S. A. **26**, 708–712 (1940). [MF 3371]

In this paper J. W. Calkin continues [cf. Trans. Amer. Math. Soc. **45**, 369–442 (1939)] to investigate problems of the general theory of linear transformations in the abstract Hilbert space $\mathfrak{H}$ which are analogous to boundary value problems. He now considers only those more special problems which are associated with a semi-bounded symmetric form; this enables him to provide a more complete analysis and to treat problems and applications to differential operators in a considerably simpler way than formerly. The present paper gives only a sketch of the method and of the principal results; a detailed discussion will appear later.

The investigation is based upon a fundamental result of K. Friedrichs [Math. Ann. **109**, 465–487 and 685–713 (1934)] that to any bilinear non-negative symmetric closed form $G(f, g)$, defined for every pair of elements f, g in a dense linear subset $\mathfrak{G}$ of $\mathfrak{H}$, there is associated uniquely a self-adjoint transformation $S=S(G)$ with domain $\mathfrak{D}(S)\subseteqq\mathfrak{G}$ such that $(f, Sg)=G(f, g)$ for all f in $\mathfrak{G}$ and g in $\mathfrak{D}(S)$. A closed linear transformation B from $\mathfrak{G}$ to a unitary or Hilbert space $\mathfrak{N}$ is called a "boundary operator on $\mathfrak{G}$" if its domain $\mathfrak{D}(B)$ and its range $\mathfrak{R}(B)$ are dense in $\mathfrak{G}$ and $\mathfrak{N}$, respectively, and if the solutions of $Bf=0$ in $\mathfrak{D}(B)$ are dense in $\mathfrak{H}$; this is the fundamental abstraction of the notion of boundary values. Then by application of Friedrichs' theorem the following basic theorem can be stated: There exists one and only one pair of transformations T, N with the same subset of $\mathfrak{D}(B)$ as domain and with subsets of $\mathfrak{H}$, $\mathfrak{N}$, respectively, as range, such that the transformation of all vectors $\{f, Bf\}$ into the vectors $\{Tf, Nf\}$ is a self-adjoint transformation of the Hilbert space $\mathfrak{H}\oplus\mathfrak{N}$, and such that the identity

$$(f, Tg)+(Bf, Ng)=G(f, g)$$

holds for all f in $\mathfrak{D}(B)$ and all g in $\mathfrak{D}(T)$. This identity obviously has the same role as Green's formulas do for boundary problems of differential equations. Some concrete examples are given briefly; the simplest boundary problem for an ordinary differential equation of second order comes out for

$$G(f, g)=\int_a^b (pf'\bar{g}'+qf\bar{g})dx,$$

$\mathfrak{N}$ a two-dimensional unitary space and $Bf=\{f(a), f(b)\}$. Furthermore, the author indicates the connections with the theory of his preceding paper; he emphasizes rather less briefly some groups of problems which can be interpreted and treated only by the concepts of the new theory; and he states some of his results in that respect which will be proved in the projected comprehensive discussion, and of which we mention only the formulation of analogues to Dirichlet's problem and Dirichlet's principle for those general boundary operators. *E. D. Hellinger.*

Powzner, A. Über positive Funktionen auf einer Abelschen Gruppe. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 294–295 (1940). [MF 3328]

A terse proof for the theorem that a continuous positive definite function $f(x)$ on a compact separable Abelian group is an integral $\int e^{ix(y)}d\varphi(x)$, where $\varphi(x)$ is an absolutely additive measure function on the space of characters. The author does not emphasize how the restriction of $f(x)$ being continuous is reflected in the assertion or proof of the theorem. *S. Bochner* (Princeton, N. J.).

Raikov, D. Positive definite functions on commutative groups with an invariant measure. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 296–300 (1940). [MF 3329]

Generalizing the reviewer's formula

$$\varphi(g)=\int_{-\infty}^{\infty} e^{ixg}d\sigma(x), \qquad d\sigma(x)\geqq 0,$$

for any continuous positive definite function on the straight line, the author shows that a measurable positive definite function $\varphi(g)$ on a commutative group with Haar measure is almost everywhere equal to $\int x(g)d\sigma(x)$, where $\Delta\sigma(x)$ is one of a certain family of non-negative set-functions of bounded variation on the locally bicompact space of all continuous characters of the group. *S. Bochner.*

Raikov, D. Positive definite functions on discrete commutative groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 324–328 (1940). [MF 3247]

The positive definite functions $f(g)$ on a group G (no topology on G is assumed and no continuity of $f(g)$ is required) represent in a one-to-one way the positive functionals L of the normed vector lattice of (real) almost periodic functions $g(x)$ on the group of characters; with $\|g(x)\|=\text{l.u.b.}|g(x)|$ and $\|L\|=f(0)$. *S. Bochner.*

Kawada, Yukiyosi and Itô, Kiyosi. On the probability distribution on a compact group. I. Proc. Phys.-Math. Soc. Japan (3) **22**, 977–998 (1940). [MF 3796]

A probability distribution on a compact separable group G is a non-negative set-function of total value 1 which is defined on each Borel set. It is a positive functional on the space of almost periodic functions and as such has a Fourier expansion in representation coefficients. The authors first verify that a sequence of such functionals is convergent if and only if their expansions are convergent term by term, and that the composition (convolution) of two probability distributions corresponds to multiplication of its representation matrices. A distribution $p(E)$ is stable, that is, idempotent, if and only if it is proportional to the Haar measure of a closed subgroup H of G and 0 outside; and in every case the limit, as $n\to\infty$, of $n^{-1}(p^{(1)}(E)+\cdots+p^{(n)}(E))$, where $p^{(n)}(E)$ is the nth power of $p(E)$, leads to a stable distribution. The authors also have a statement if $p^{(n)}(E)$ is not necessarily the nth iterate of $p(E)$. *S. Bochner.*

Khourguine, J. et Tschetinine, N. Sur les sous-anneaux fermés de l'anneau des fonctions à n dérivées continues. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 288–291 (1940). [MF 3891]

Call a set of functions a ring R if when $x, y\in R$ then $x+y$, xy, $kx\in R$ (k real). We owe to Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)] the elegant generalization of the Weierstrass polynomial approximation theorem, according to which a ring R of continuous real functions over a bicompact space H will be everywhere dense in the space of all real continuous functions on H if and only if (a) for $t_1\in H$, $x(t_1)\neq 0$ for some $x(t)\in R$, and (b) for $t_1\neq t_2\in H$, $x(t_1)\neq x(t_2)$ for some $x(t)\in R$. The authors of this note consider the special

case of $H=I(0\leqq t\leqq 1)$. Let D_n be the space of real functions with n continuous derivatives on I, in which convergence means uniform convergence of all n derivatives. The following analogue of Stone's theorem is proved: In order that R, a closed ring in D_n, shall equal D_n, the following conditions are necessary and sufficient: (a') $x(t)\equiv 1\in R$; (b') the same as (b); (c') for $t_1\in I$, $x'(t_1)\neq 0$ for some $x(t)\in R$.

J. A. Clarkson (Philadelphia, Pa.).

Šilov, G. Sur la théorie des idéaux dans les anneaux normés de fonctions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 900–903 (1940). [MF 3583]

The author considers a ring R of continuous functions, supposed normed, and satisfying certain additional conditions. If F is any closed set in the base space, then $I(F)$, the set of all functions of R which vanish on F, is a closed ideal of R; R is said to be of type N if this correspondence is one-to-one. A (formally) less restrictive condition is that the maximal ideals of R are just those closed ideals formed from all those functions which vanish at a fixed point; such rings are said to be of type N^*. Conditions are given, without proof, which are necessary and sufficient for a ring R to be of each type. The author's theorems do not suffice to decide whether the rings of type N are a proper subset of those of type N^*. *J. A. Clarkson* (Philadelphia, Pa.).

Lorch, Edgar R. The integral representation of weakly almost-periodic transformations in reflexive vector spaces. Trans. Amer. Math. Soc. **49**, 18–40 (1941). [MF 3666]

A transformation V in a reflexive, that is, regular, normal linear space $\mathfrak{B}$ is said to be weakly almost-periodic if it has a bounded inverse V^{-1} and if the set $V^n f$ $(n=0, \pm 1, \pm 2, \cdots)$ of iterated transformations is weakly conditionally compact for each f in $\mathfrak{B}$. Included in this class of transformations are all rotations as well as other transformations which heretofore have not been studied even in Hilbert space. An analysis of the structure of weakly almost-periodic transformations shows that for $-\infty<\lambda<\infty$ there is a pair of closed linear manifolds $\{\mathfrak{E}_\lambda, \mathfrak{F}_\lambda\}$ which plays here the role of a resolution of the identity. These manifold pairs reduce V and enable one in a certain sense to obtain an integral representation for V. For weakly almost periodic transformations V the spectrum consists of the point spectrum, the continuous spectrum and the resolvent set, and these are all the possibilities. There is a quite remarkable relationship between the spectrum of V and the manifold sets $\{\mathfrak{E}_\lambda, \mathfrak{F}_\lambda\}$ which is quite comparable to the relation between a resolution of the identity and the spectrum of a self-adjoint transformation in Hilbert space. The paper closes with a thorough study of the transformation $H=-i(V-I)(V+I)^{-1}$.

H. H. Goldstine (Ann Arbor, Mich.).

Calkin, J. W. Symmetric transformations in Hilbert space. Duke Math. J. **7**, 504–508 (1940). [MF 3416]

If a symmetric transformation is such that $(H+\lambda I)^{-1}$ exists and is bounded for some real λ, it possesses self-adjoint extensions. This note also discusses the construction of these extensions. *F. J. Murray* (New York, N. Y.).

Eidelheit, M. On isomorphisms of rings of linear operators. Studia Math. **9**, 97–105 (1940). (English. Ukrainian summary) [MF 3483]

Let E be a Banach space and $\mathfrak{A}(E)$ be a linear ring of linear operations whose domain is E and range in E. If $U(x)\in\mathfrak{A}(E)$ and norm $|U|$ of U is defined by $|U|=\sup_{|x|\leq 1}|U(x)|$, then $\mathfrak{A}(E)$ becomes a Banach space in which the norm satisfies the additional condition $|U_1U_2|\leqq|U_1||U_2|$. The author proves that, if $|U|^*$ is another norm in $\mathfrak{A}(E)$ satisfying this additional condition and is such that $\mathfrak{A}(E)$ is a complete space relative to this norm, then the norms $|U|$, $|U|^*$ are equivalent. He further proves that two rings $\mathfrak{A}(E_1)$, $\mathfrak{A}(E_2)$ are algebraically isomorphic if and only if the corresponding spaces E_1 and E_2 are isomorphic (that is, there exists 1-1 linear transformation of E_1 into E_2). Furthermore, if $V=\Phi(U)$ is the isomorphism between $\mathfrak{A}(E_1)$ and $\mathfrak{A}(E_2)$, Φ being 1-1, additive and multiplicative, then there exists an isomorphism $y=A(x)$ between E_1 and E_2 such that $\Phi(U)=AUA^{-1}$ for $U\in\mathfrak{A}(E_1)$. The same result holds if $\Phi(U)$ is assumed to be multiplicative and continuous and E_1 is at least two-dimensional. *J. D. Tamarkin* (Providence, R. I.).

Neumark, M. On the square of a closed symmetric operator. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 866–870 (1940). [MF 3581]

Neumark, M. A complement to the paper "On the square of a closed symmetric operator." C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 207–208 (1940). [MF 3602]

In these notes the author constructs a symmetric operator H such that the domain of H^2 is (0). If $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are the deficiency manifolds for H, it is shown that

$$[D(H^2)]=\mathfrak{H}\ominus[D(H^*\cdot H)\cdot[\mathfrak{M}^+ + \mathfrak{M}^-]],$$

where the brackets denote the closure. In terms of multivalued operators, it is shown that $H^{2*}=[H^{*2}]$. If $\mathfrak{M}^+ + \mathfrak{M}^-$ is closed, $D(H^2)=\mathfrak{H}$. *F. J. Murray* (New York, N. Y.).

Neumark, M. Deficiency spaces of the direct product of symmetric operators. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 209–210 (1940). [MF 3603]

The direct product $\mathfrak{H}_1\otimes\mathfrak{H}_2$ of two Hilbert spaces is the space formed by the use of the distributive operator $\otimes$. If H_1 and H_2 are symmetric with Cayley transforms V_1 and V_2, the Cayley transform of $H_1\otimes H_2$ is not $V_1\otimes V_2$. In a previous paper [Bull. Acad. Sci. URSS [Izvestia] **1939**, 274–278], the author obtained an integral expression for the projection on the deficiency manifolds of $H_1\otimes H_2$, if H_1 is self-adjoint. Here this restriction is removed.

F. J. Murray (New York, N. Y.).

Gantmacher, Vera. Über schwache totalstetige Operatoren. Rec. Math. [Mat. Sbornik] N.S. **7 (49)**, 301–308 (1940). (German. Russian summary) [MF 2791]

A linear operator in a Banach space E is said to be weakly completely continuous in case it takes bounded sets into weakly compact sets. The author proves (i) A is weakly c.c. if and only if the conjugate operator A^* is weakly c.c., (ii) if A is weakly c.c. and $f_n\to f$ weakly as functions, then $A^*f_n\to A^*f$ weakly as elements, (iii) for a weakly c.c. A we have $f_n(x)=c_n$ solvable for x in case

$$|\sum_1^n \lambda_i c_i|\leqq M|\sum_1^n \lambda_i A^* f_i|,$$

and, conversely, when the moment theorem holds A is weakly c.c. For a subspace E_1 of $\overline{E}$ let $\bar{E}_1$ consist of those $F_0\in\overline{E}$ for which $\underline{\lim}\, F_n(f)\leqq F_0(f)\leqq\overline{\lim}\, F_n(f)$ for all $f\in\overline{E}$ and some $F_n\in E_1$. Then A is weakly c.c. if and only if $A^{**}(\bar{E}_1)\subset E_1$, where E_1 is the set in $\overline{E}$ determined by E. Several other properties of and criteria for weakly c.c. operators are given. *N. Dunford* (New Haven, Conn.).

Yosida, Kôsaku. On the theory of spectra. Proc. Imp. Acad. Tokyo **16**, 378–383 (1940). [MF 3364]

A purely algebraic approach to the spectral theory for unbounded operators is developed and at the same time an algebraic characterization of the ring of all real Borel functions is obtained. The methods used are different from those of M. H. Stone, S. Kakutani, I. Gelfond and M. and S. Krein, who have discussed similar problems, in that neither the ideals nor the homomorphisms are used explicitly in the fundamental way they were by these authors. The algebraic system which leads to the spectral theory is a commutative and associative ring with a unit, having the real numbers as distributive operators, and in which $X^2=0$ implies $X=0$. The rest of the axioms concern a partial ordering $X>Y$ which is defined to mean that there is a $Z\neq 0$ such that $Z^2=X-Y$. About this ordering it is assumed that (i) $X>0$, $Y\geqq 0$ imply $X+Y>0$, (ii) $X^+=\sup(X,0)$ exists and $X^+(-X)^+=0$, and (iii) a bounded monotone increasing sequence has a least upper bound and, if $A>0$, $X_{n+1}\geqq X_n\geqq 0$, then $A\sup X_n=\sup AX_n$, provided X_n is bounded. These axioms suffice for a development of the spectral theory but must be augmented by (iv) $\inf(|X_i|,|X_j|)=0$ $(i\neq j)$ (where $|X|=X^++(-X)^+$) implies

$$\sup_{n\geq 1}\inf_{m\leq n}\sum_{i=1}^{m}X_i=\inf_{n\geq 1}\sup_{m\leq n}\sum_{i=1}^{m}X_i,$$

in order to obtain an isomorphism with the ring of all real Borel functions on a totally disconnected bicompact space. In this isomorphism $X\leftrightarrow f_X(t)$ the operations of the ring and scalar multiplication correspond to the usual ones on functions, whereas the notion $X>0$ is equivalent to the statement $f_X(t)>0$ except on a set of first category.

N. Dunford (New Haven, Conn.).

Bebutoff, M. Sur les systèmes dynamiques dans l'espace des fonctions continues. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 904–906 (1940). [MF 3584]

Let R_u denote the space of continuous real functions on $-\infty<x<+\infty$ metrized so that convergence means uniform convergence on every finite interval. The function $f_0(\varphi(x),t)=\varphi(x+t)$ defines a particular continuous flow M_u in R_u. Let $f(p,t)$ define a continuous flow M in a topological space R. Any continuous real-valued function $\Phi(p)$ on R defines a continuous map Ψ of R into R_u (namely, $\Psi(p)=\Phi[f(p,x)]$) such that the flow M is mapped continuously into the flow M_u. The question is: when can this process be made to yield a topological embedding of M in M_u? In other words, when does there exist a function Φ such that Ψ is a homeomorphism of R onto $\Psi(R)$? It is asserted that a sufficient condition is that R be a compact metric space and that the flow M have at most one fixed point, and this result is generalized to locally compact Hausdorff spaces with denumerable base. Other results are concerned with generalizations of the notions of minimal set and stability in the sense of Poisson. All results are stated without proof.

J. C. Oxtoby (Bryn Mawr, Pa.).

Calculus of Variations

Santalo, L. A. On some geometrical problems concerning aviation. Bol. Mat. **13**, 66–71 (1940). (Spanish) [MF 2209]

Geometrical solutions of some extremal problems occurring in aviation. *A. González Domínguez* (Buenos Aires).

Busemann, Herbert and Mayer, Walther. On the foundations of calculus of variations. Trans. Amer. Math. Soc. **49**, 173–198 (1941). [MF 3920]

While existence theorems for the problem $\int F(x,\dot{x})dt=\min.$ can be established without assuming the differentiability of F, in establishing differentiability properties of the minimizing curve it has been traditional to use the Euler equations. In this paper theorems on differentiability of the solution are proved without assuming the existence of partial derivatives of F. The principal theorem is that in plane problems, if F is positive definite and locally Lipschitzian, and for each x the set $\Gamma(x)$ of points $\dot{x}$ such that $F(x,\dot{x})\leqq 1$ is strictly convex, the minimizing curve is continuously differentiable. The methods of proof are necessarily quite different from the traditional methods. The normals to supporting planes of $\Gamma(x)$ replace the vector $(\partial F/\partial\dot{x}_i)$. In the plane, arcs tangential to certain fields of these vectors are called pseudo-circles. A fundamental part of the argument is that the integral along an arc of a pseudo-circle is less than the integral along any curve with the same ends but otherwise outside the pseudo-circle. By examples it is shown that the hypotheses of the principal theorem do not ensure the existence of second derivatives; moreover, through a given point and in a given direction there may be no minimizing curve, or there may be more than one. *E. J. McShane* (Charlottesville, Va.).

Boerner, Hermann. Variationsrechnung aus dem Stokesschen Satz. Math. Z. **46**, 709–719 (1940). [MF 3377]

The author gives a unified treatment of certain of the standard necessary and sufficient conditions for a curve $x_i=x_i(t)$ which is to minimize an integral $\int f(t,x_i,\dot{x}_i)dt$. Consistent use is made of the generalized Stokes theorem in the proofs. The necessary conditions of Euler and Legendre and the necessary transversality conditions are derived first. Geodesic fields are then defined and the sufficient condition of Weierstrass and the transversality theorem of Kneser are proved. This paper serves as an introduction to the paper reviewed below. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Boerner, Hermann. Über die Legendresche Bedingung und die Feldtheorien in der Variationsrechnung der mehrfachen Integrale. Math. Z. **46**, 720–742 (1940). [MF 3378]

In this paper the author generalizes the results of the paper reviewed above to manifolds $x_i=x_i(t_1,\cdots,t_n)$ which are to minimize the multiple integral

$$\int\cdots\int f(t_\alpha,x_i,\partial x_i/\partial t_\alpha)dt_1\cdots dt_n.$$

The author is able to give an elegant unified treatment of the various field theories of Carathéodory, DeDonder, Weyl and Lepage (for sufficient conditions), to give elegant proofs of the necessary conditions of Euler and Hadamard-Legendre, and to develop an apparently new formulation of the transversality conditions by means of his extensive use of the generalized Stokes theorem. Of special interest is the author's treatment of problems with variable boundary values and the necessary and sufficient conditions which he obtains for such problems. He gives an example to show that a surface which satisfies the Hadamard sufficient Legendre condition (for problems with fixed boundary) and the transversality conditions is not necessarily a minimizing surface for a problem with variable boundary values.

C. B. Morrey, Jr. (Berkeley, Calif.).

McShane, E. J. Necessary conditions in generalized-curve problems of the calculus of variations. Duke Math. J. 7, 1–27 (1940). [MF 3387]

The present paper is the second of three papers whose chief purpose is to establish general existence theorems for the problem of Bolza. In the first paper [Duke Math. J. 6, 513–536 (1940); these Rev. 2, 59] the author developed the theory of generalized curves, originated by L. C. Young. The purpose of the present paper is to develop the theory of the calculus of variations for generalized curves, and in particular to develop necessary conditions for a minimum for the problem of Bolza. No normality assumptions are made. The author proves simultaneously analogues of the multiplier rule, the transversality condition and the necessary conditions of Weierstrass and Clebsch. The proofs are made in such a way that they are valid also if one restricts the admissible curves to be ordinary rectifiable curves. A theory of supporting sets is developed, as well as an analogue of the Dresden corner condition. The case when isoperimetric side conditions appear is also treated. In an appendix it is proved that if a function on a closed set can be extended locally to be of class $C^{(m)}$, it can be extended so as to be of class $C^{(m)}$ over the whole space. In fact a more general result than this is established [cf. the following review]. *M. R. Hestenes* (Chicago, Ill.).

McShane, E. J. Existence theorems for Bolza problems in the calculus of variations. Duke Math. J. 7, 28–61 (1940). [MF 3388]

The present paper is a continuation of the paper reviewed above. Its purpose is to find conditions under which the minimizing curves are curves in the ordinary sense and not generalized curves. Three very general existence theorems are given for the parametric problem of Bolza. The comparison curves are the totality of generalized curves lying on a closed set E in $(y_1, \cdots, y_n)$-space in which the integrand function $f(y, r)$ and the functions $\phi^a(y, r)$ defining the differential side conditions have the usual properties that must be assumed. Granting the existence of minimizing sequence on a bounded set having uniformly bounded lengths, the minimizing curve will be an ordinary curve if one of the following criteria is essentially satisfied: (1) The multipliers belonging to the minimizing curve can be chosen so that certain associated supporting sets have first and last portions. (2) At each point y_0 on the minimizing curve the solutions r of $\phi^a(y_0, r)=0$ form a convex set on which $f(y_0, r)$ is convex. The author gives further criteria implying these but which have the advantage that they are expressed in terms of the original data instead of in terms of the minimizing curve. When these theorems are applied to free problems, one obtains results that are apparently more general than those previously given. The author obtains, by the use of these theorems, new results for isoperimetric problems. When interpreted for the problem of Mayer these existence theorems give results that overlap with those given by Graves, but neither include the other. Finally an example is given illustrating the generality of the theorems established. *M. R. Hestenes* (Chicago, Ill.).

Mammana, Gabriele. La variazione seconda generalizzata nel caso dei punti terminali mobili e problemi sugli autovalori connessi. Ann. Mat. Pura Appl. (4) 19, 81–106 (1940). [MF 3046]

This paper treats the problem of determining conditions under which the functional

$$J[y]=\int_{x_1}^{x_2}[a(x)y'^2+2b(x)yy'+c(x)y^2 +2p(x)y'+2q(x)y+r(x)]dx,$$

$a(x)>0$ on $x_1\leqq x\leqq x_2$, admits a minimum in the class of functions $y=y(x)$ of class C' on this interval; the author also considers certain boundary problems associated with the related differential equation $(az')'+\rho(b'-c)z=0$. Some of the boundary conditions considered with this differential equation are linear and two-point in character, whereas others are of the form of equalities or inequalities satisfied by the end-form $[z(az'+\rho bz)]_{x_1}^{x_2}$. Although the author considers specifically as a special instance of his discussion the case when $b'\equiv c$ on x_1x_2, many of his stated results on the boundary problems considered do not hold in this case. In particular, by simple examples it may be shown that conclusion (b) of Theorem I, as well as Theorems II, III and VII, are not valid in general if $b'\equiv c$. If $b'\not\equiv c$ on x_1x_2, some of the principal results on boundary problems given in Theorem III could have been deduced as direct consequences of the Sturmian theory [see, for example, Ince, Ordinary Differential Equations, 1927, Ch. X], and the remainder are deducible from the Sturmian theory and slight additional argument. As far as the conditions given for the functional $J[y]$ to possess a minimum are concerned, the necessary conditions obtained are well known in the calculus of variations, whereas the sufficient conditions presented could have been obtained readily from equally well-known results of the calculus of variations supplemented by a very simple compatibility theorem for the linear Euler equation of the problem under consideration. *W. T. Reid* (Chicago, Ill.).

Morse, Marston and Tompkins, C. B. Unstable minimal surfaces of higher topological types. Proc. Nat. Acad. Sci. U. S. A. 26, 713–716 (1940). [MF 3372]

Let g_0, g_1 be rectifiable simple closed curves in n-space separated by a hyperplane and satisfying the "chord arc" condition; that is, the ratio of the length of an arbitrary chord to the length of the shorter of the corresponding arcs is bounded from zero. If $g_s(t)$, $0\leqq t\leqq 2\pi$, is the representation of g_s $(s=0, 1)$ with t proportional to arc length, other representations have the form $g_s(h(\alpha))$, where the transformation $h(\alpha)$ is non-decreasing and $h-2\pi\alpha$ has period 2π. The space II is the space of sets $(\varphi_0, \varphi_1, h_0, h_1, \rho)$, where $g_0(h_0(\alpha))$ represents g_0 on the circle $r=1$, $g_1(h_1(\alpha))$ represents g_1 on $r=\rho$, the transformations $\varphi_s(\alpha)$ are continuous and leave three chosen values of α invariant, and h_s either is degenerate or has the form $h_s=\varphi_s(T(\alpha))$, where $T(\alpha)$ arises from a one-to-one conformal map of the unit circle on itself. In II a metric is introduced, and a function $W(P)$ is defined in terms of the Douglas functional and the Dirichlet integral of the vector function harmonic in $\rho<r<1$ and assuming the boundary values $g_s(h_s(\alpha))$. It is announced that this space and functional serve better than the Dirichlet functional in investigating unstable minimal surfaces. The subset $W<\infty$ has connectivities $(1, 0, 0, \cdots)$. Homotopic critical points are differentially critical. The following theorem is announced. If g_0 and g_1 bound a ring minimal surface belonging to a minimizing set of critical points of W, there either exists a ring minimal surface of non-minimizing type bounded by g_0 and g_1, or else a disc minimal surface of non-minimizing type bounded by g_0 or by g_1.

E. J. McShane (Charlottesville, Va.).

Morse, Marston and Tompkins, C. Minimal surfaces not of minimum type by a new mode of approximation. Ann. of Math. (2) **42**, 62–72 (1941). [MF 3674]

The authors consider a function $f^{(0)}(\varphi)$ which is defined and is the pointwise limit of a sequence $\{f^{(n)}(\varphi)\}$ of functions defined on a metric space M. The functions $f^{(0)}$ and $f^{(n)}$ satisfy (individually and collectively) certain hypotheses which enable them to apply their previous theories of critical points to the $f^{(n)}$ and to define those of $f^{(0)}$ in a significant way in terms of those of the $f^{(n)}$. A set A is said to be minimizing relative to $f^{(0)}$ if $f^{(0)}=$a constant c on A and if there is an ϵ-neighborhood A_ϵ of A such that $f^{(0)}>c$ on $A_\epsilon-A$; if φ belongs to such a set A it is a critical point for $f^{(0)}(\varphi)$. A critical point φ of $f^{(0)}$ is said to be of non-minimizing type if every neighborhood of φ contains a ψ for which $f(\psi)<f(\varphi)$. The principal result of this part of the paper is that, if $\varphi_1 \epsilon A_1$ and $\varphi_2 \epsilon A_2$, A_1 and A_2 being disjoint minimizing sets for $f^{(0)}$, then there exists a critical point φ_0 for $f^{(0)}$ which is of non-minimizing type.

The above result is translated into a very general result concerning the existence of minimal surfaces of non-minimizing type as follows: Let $g^{(0)}: \mathbf{x}=\mathbf{x}(s)$ ($s=$arc length) be a simple closed curve (of length 2π, for convenience) such that $\overset{\frown}{P_1P_2}/\overline{P_1P_2}$ is uniformly bounded for all P_1, P_2 on $g^{(0)}$, $\overset{\frown}{P_1P_2}$ being the length of the shorter arc. Let $g^{(n)}: \mathbf{x}=\mathbf{x}_n(s)$ be a sequence of similar curves in which the $\mathbf{x}_n(s)$ are of class C'' and converge uniformly to $\mathbf{x}(s)$ for $0\leqq s\leqq 2\pi$. The space M consists of all continuous functions $\varphi(\theta)$ such that $\varphi(\theta+2\pi)=\varphi(\theta)+2\pi$, $\varphi(\theta_i)=\theta_i$ for three fixed distinct values θ_1, θ_2, θ_3 with $0\leqq\theta_i<2\pi$, and the Douglas functional for $\mathbf{x}[\varphi(\theta)]$ is finite. The function $f^{(0)}(\varphi)$ is this Douglas functional for $\mathbf{x}[\varphi(\theta)]$ and the $f^{(n)}(\varphi)$ are the same for $\mathbf{x}_n[\varphi(\theta)]$. The distance (φ_1, φ_2) in M is $\max_{0\leq\theta\leq 2\pi}|\varphi_1(\theta)-\varphi_2(\theta)|$. It is finally shown that if φ is a critical point for $f^{(0)}$, then the potential vector with boundary values $\mathbf{x}[\varphi(\theta)]$ defines a minimal surface. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Morse, Marston and Tompkins, C. Corrections to our paper on the existence of minimal surfaces of general critical types. Ann. of Math. (2) **42**, 331 (1941). [MF 3689]

The paper appeared in Ann. of Math. (2) **40**, 443–472 (1939).

Courant, Richard. Critical points and unstable minimal surfaces. Proc. Nat. Acad. Sci. U. S. A. **27**, 51–57 (1941). [MF 3652]

The author discusses minimal surfaces (homeomorphic to a circular disc) of general critical type which are bounded by a polygon Γ with $N+3$ vertices. It is shown that the variational problem is equivalent to that of finding the stationary values of an ordinary differentiable function of N variables. It follows that the type number of any minimal surface bounded by Γ cannot exceed $N-1$. To show this equivalence, the author considers vector functions $\mathfrak{X}(u, v)$ defined and of class D' (continuous with piecewise continuous derivatives) in the upper half plane which map $v=0$ into Γ, $u=\infty$, $u=-1$ and $u=0$ corresponding to three consecutive vertices on Γ. For any such map, the remaining vertices correspond in order to points $u_1, \cdots, u_N$ on $v=0$ with $0<u_1<u_2<\cdots<u_N$. For each such set of numbers u_i, the author constructs (as in his solution of the Plateau problem) the harmonic vector which minimizes the Dirichlet functional among all admissible vector functions in which the vertices of Γ correspond to the u_i as above. It is shown that for each set $(u_1, \cdots, u_N)$, the above vector is unique. The author calls the N-parameter family of these vectors $\mathfrak{M}$ and defines the function $d(u_1, \cdots, u_N)$ as the value of the Dirichlet functional for the minimal vector. It is then shown that any harmonic vector yielding a minimal surface bounded by Γ is in $\mathfrak{M}$ and also that $d(u_1, \cdots, u_N)$ possesses continuous first derivatives and becomes infinite as the point $(u_1, \cdots, u_N)$ approaches the boundary of its domain $(0<u_1<u_2<\cdots<u_N)$. The minimal surfaces correspond to the critical points of $d(u_1, \cdots, u_N)$. *C. B. Morrey, Jr.*

Theory of Probability

Broderick, T. S. and Schrödinger, E. Boolean algebra and probability theory. Proc. Roy. Irish Acad. Sect. A. **46**, 103–112 (1940). [MF 4142]

In this note the well-known elements of Boolean algebra in its application to propositions and to sets, and the notions of set-functions and characteristic function are assembled, and with the aid of a convenient symbolic notation are applied to elementary problems of probability, such as calculating the expected number of trials needed for "completion" in a succession of independent trials of an experiment having n possible disjoint outcomes (that is, in order that each occur at least once). The symbolic method is the only original contribution. *B. O. Koopman.*

Reichenbach, Hans. Note on probability implication. Bull. Amer. Math. Soc. **47**, 265–267 (1941). [MF 4172]

In the axiom system of his book "Wahrscheinlichkeitslehre," Reichenbach postulated that the probability implication $O \supset_p P$, where O is an empty class, should hold for an ambiguous p, in order to relate his probability implication to Russell's material implication. J. C. C. McKinsey [Bull. Amer. Math. Soc. **45**, 799–800 (1939); these Rev. **1**, 61] pointed out that this ambiguous p is not restricted to the range (0, 1), contradicting the restriction of the probability degree p in general to that range. In the present note Reichenbach acknowledges the validity of McKinsey's criticism, and chooses to avoid the contradiction by removing the restriction of p to the range (0, 1) for the case that O is empty. *S. C. Kleene* (Madison, Wis.).

Koopman, B. O. Intuitive probabilities and sequences. Ann. of Math. (2) **42**, 169–187 (1941). [MF 3679]

The author continues his formalization of an individual's concept of probability ("intuitive probability") [Ann. of Math. (2) **42**, 269–292 (1940), these Rev. **1**, 245; and Bull. Amer. Math. Soc. **46**, 763–774 (1940), these Rev. **2**, 106]. The concept of a sequence of independent similar events is formalized in such a way that the probability of the event can be numerically evaluated (not true of all events in the author's theory) and the probability becomes the limit of the usual success ratios. *J. L. Doob* (Urbana, Ill.).

Ferrari, Esther. On a paradox of Bertrand. Revista Union Mat. Argentina **7**, 1–6 (1940). (Spanish) [MF 3854]

Selberg, Henrik L. Zwei Ungleichungen zur Ergänzung des Tchebycheffschen Lemmas. Skand. Aktuarietidskr. **1940**, 121–125 (1940). [MF 3931]

Let $f(x)$ be a frequency function with vanishing first moment and a finite second moment σ^2. Putting $W(-\alpha, \beta)$

$=\int_{-a}^{\beta} f(x)dx$, it is shown that $W(-\alpha, \beta) \geqq \alpha^2\{\alpha^2+\sigma^2\}^{-1}$ if $\alpha(\beta-\alpha) \geqq 2\sigma^2$, and $W(-\alpha, \beta) \geqq \{4\alpha\beta-4\sigma^2\}\{\alpha+\beta\}^{-2}$ if $\alpha(\beta-\alpha) \leqq 2\sigma^2$. *W. Feller* (Providence, R. I.).

Selberg, Henrik L. Über eine Ungleichung der mathematischen Statistik. Skand. Aktuarietidskr. **1940**, 114–120 (1940). [MF 3930]

Suppose that the function $f(x)$ of the preceding review is non-decreasing for $x<x_0$ and non-increasing for $x>x_0$. It is shown that $W(-a, a) \geqq 1-\theta\sigma^2/a^2$, where θ satisfies $\theta^3-9\theta^2+3\theta+1=0$ and $0.56<\theta<0.57$. *W. Feller.*

Greenwood, Joseph A. The first four moments of a general matching problem. Ann. Eugenics **10**, 290–292 (1940). [MF 3207]

The author considers two decks of cards, a and b, each containing n cards and t suits. Deck a contains a_i cards of the ith suit and deck b contains b_i cards of the ith suit. A card is drawn at random from each deck. If the two cards are of the same suit, this constitutes a matching. The drawing is continued without replacing until the two decks are exhausted. A distribution function for the number of matchings is thus defined. The author gives formulas for the first four moments of this distribution. *A. H. Copeland.*

Brown, B. H. Simple examples of limiting processes in probability. Amer. Math. Monthly **48**, 98–102 (1941). [MF 3912]

Mood, A. M. The distribution theory of runs. Ann. Math. Statistics **11**, 367–392 (1940). [MF 3417]

Consider random arrangements of n elements of two kinds, say a's and b's, and let r_i denote the number of runs of a's of length i; for a given $k<n$ let $s_j=r_j$ if $j<k$ and $s_k=r_k+r_{k+1}+\cdots$. Denote the corresponding variables for the b's by ρ_i and σ_k. The total number m of the a's may either be known ($m<n$) or else we may be concerned with a random sample, of size n, of a binomial population. The author determines in either case the distribution of the random variables r_i and s_k; he finds the corresponding moments and proves that the joint distribution of $(s_1, \cdots, s_k)$ as well as of $(s_1, \cdots, s_k, \sigma_1, \cdots, \sigma_l)$ is, in the limit for $n\to\infty$, of the normal type. The corresponding results are also deduced for the case of more than two kinds of elements. *W. Feller* (Providence, R. I.).

Norris, Nilan. The standard errors of the geometric and harmonic means and their application to index numbers. Ann. Math. Statistics **11**, 445–448 (1940). [MF 3422]

Let $x_1, x_2, \cdots$ be mutually independent chance variables with a common distribution function. The author shows, applying a method of the reviewer [Ann. Math. Statistics **4**, 160–169 (1935)], that, if the given distribution function satisfies simple conditions, then the harmonic and geometric means $1/\sum_1^n 1/x_j$, $\Pi_1^n x_j^{1/n}$ are approximately normally distributed, for large n, with explicitly given standard deviations. *J. L. Doob* (Urbana, Ill.).

Consoli, T. Généralisation d'un théorème sur la probabilité de la somme d'un nombre infini de variables aléatoires. Rev. Fac. Sci. Univ. Istanbul (A) **5**, 1–17 (1940). (French. Turkish summary) [MF 3240]

The author considers the distribution of a sum of independent discrete random variables. He gives conditions sufficient to insure that a certain function linearly related to this distribution will approach the probability density $(1/\sqrt{\pi})e^{-v^2}$ as the number of terms in the sum is indefinitely increased. His result includes as special cases a theorem proved by von Mises [Rev. Fac. Sci. Univ. Istanbul (N.S.) **1**, 61–80 (1935)] and a theorem proved by Bawly [ibid. **2**, 79–92 (1937)]. *A. H. Copeland* (Ann Arbor, Mich.).

Geiringer, Hilda. A generalization of the law of large numbers. Ann. Math. Statistics **11**, 393–401 (1940). [MF 3418]

This article is concerned with a law of large numbers for continuous statistical functions. Such a function assigns a value to the partition of the first n terms of a sequence of uniformly bounded random variables, a partition of n variables being a function $S_n(x)$ such that $n\cdot S_n(x)$ is the number of these variables not exceeding x. The same statistical function also assigns a value to the average of the distributions of these n variables. The probability that the difference between these two values will exceed a given positive number ϵ is shown to approach zero as n is indefinitely increased provided the random variables satisfy a certain condition restricting asymptotically the degree of their dependence. *A. H. Copeland* (Ann Arbor, Mich.).

Hartman, Philip and Wintner, Aurel. On the law of the iterated logarithm. Amer. J. Math. **63**, 169–176 (1941). [MF 3640]

Let $\{x_n(t)\}$ be a sequence of independent functions defined on (0, 1) and let $\{\sigma_n(x)\}$ denote the sequence of their distribution functions. Let $\gamma_k=\int_{-\infty}^{+\infty} x^2 d\sigma_k(x)$, $\beta_n=\sum_{k=1}^n \gamma_k$ and assume that $\int_{-\infty}^{+\infty} x d\sigma_n(x)=0$ for every n. The authors prove that, if $\beta_n/n>\text{const.}>0$ and a distribution function $\tau(x)$ exists such that $\int_{|x|\geq\tau} d\sigma_n(x)=O(\int_{|x|\geq\tau} d\tau(x))$ $(\tau\to\infty)$ uniformly in n, then

$$\limsup_{n\to\infty} \Big(\sum_{k=1}^n x_k(t)\Big)/(2\beta_n \log\log \beta_n)^{\frac{1}{2}}=1$$

for almost all t. The proof makes use of the well-known "iterated logarithm" theorem of Kolmogoroff [Math. Ann. **101**, 126–135 (1929)]. *M. Kac* (Ithaca, N. Y.).

Berry, Andrew C. The accuracy of the Gaussian approximation to the sum of independent variates. Trans. Amer. Math. Soc. **49**, 122–136 (1941). [MF 3669]

Let $X_1, \cdots, X_n$ be independent random variables and let $\mu_2(X_n)$ and $\mu_3(X_k)$ denote respectively the second and third order absolute moments of X_k about its mean α_k. Suppose that $\mu_2(X_k)\neq 0$ for at least one k and put

$$\lambda(X_k)=\begin{cases}\mu_3(X_k)/\mu_2(X_k) & \text{if } \mu_2(X_k)\neq 0,\\ 0 & \text{if } \mu_2(X_k)=0.\end{cases}$$

Put furthermore $\alpha=\alpha_1+\cdots+\alpha_n$,

$$\sigma=(\mu_2(X_1)+\cdots+\mu_2(X_n))^{1/2},$$
$$G(x)=(2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}dt,$$
$$\Lambda=\max(\lambda(X_1), \cdots, \lambda(X_n))$$

and denote by $F(x)$ the distribution function of $X_1+X_2+\cdots+X_n$. The author proves then the interesting inequality that $\sup_{-\infty<x<\infty}|F(x)-G((x-\alpha)/\sigma)| \leqq 1.88\Lambda/\sigma$, which is a refinement of the central limit theorem in case of finite third order absolute moments. Various generalizations of this result are also given. *M. Kac* (Ithaca, N. Y.).

Gnedenko, B. Quelques théorèmes sur l'ensemble des puissances d'une loi de probabilité. Uchenye Zapiski Moskov. Gos. Univ. Matematika **45**, 61–71 (1940). (Russian. French summary) [MF 3732]

Let $F(x)$ be a distribution function and $f(t)$ its characteristic function. By $F^{(n)}(x)$ we understand the distribution

function whose characteristic function is $(f(t))^n$. Distribution functions $F_1(x)$ and $F_2(x)$ are said to be of the same type if numbers $b>0$ and a exist such that $F_1(bx+a)=F_2(x)$. Among several results concerning infinite divisibility, stability and domains of attraction of distribution functions the following two may be mentioned. (I) If for a certain $\mu>0$ ($\mu\neq 1$) $F(x)$ and $F^{(\mu)}(x)$ are of the same type, $F(x)$ is infinitely divisible. (II) In order that $F(x)$ be stable it is necessary and sufficient that positive numbers μ and ν exist ($\mu\neq 1$, $\nu\neq 1$) such that $\log\mu$ and $\log\nu$ are incommensurable, and $F^{(\mu)}(x)$ and $F^{(\nu)}(x)$ are of the same type.

The results and methods of proof are related to those reviewed previously [Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 61–81 (1939); these Rev. **1**, 341, where also the notions of infinite divisibility, stability and domains of attractions are defined and further references given].

M. Kac (Ithaca, N. Y.).

Agnew, R. P. and Kac, M. Translated functions and statistical independence. Bull. Amer. Math. Soc. **47**, 148–154 (1941). [MF 3832]

The following problem was due to Steinhaus: Does there exist a function $f(t)$ defined in $(-\infty, +\infty)$ such that for each sequence $\lambda_1, \lambda_2, \cdots$ of different numbers the functions $f(t+\lambda_i)$ form a sequence of statistically independent functions. The authors prove that $f(t)=\cos t^\alpha$ satisfies the condition. They also prove that if $f(t)=\cos t^m$ then each set of m different translations is a statistically independent set, but there exist $m+1$ different numbers such that the corresponding set of $m+1$ translations is not statistically independent.

P. Erdős (Philadelphia, Pa.).

Nakagami, Minoru. A method of calculating probability by means of differential operator. Nippon Elec. Commun. Engrg., no. **21**, 28–34 (1940). [MF 4286]

Instead of the characteristic functions now commonly used, the author introduces uni- and bilateral Laplace transforms (L.t.) of probability distributions (p.d.). Given the L.t., the p.d. follows from the well-known complex inversion formula: to indicate this relationship the author uses the formal symbolism of the Heaviside calculus. In the usual manner it is shown that the sum of normal variables is itself normal, and that addition of independent random variables implies multiplication of corresponding L.t. The author is led to the study of L.t. in an indirect way. He considers a function ξ of n random variables. The probability of ξ lying between R_1 and R_2 can be written in the form

$$\int\int\cdots\int H(R_1, R_2)w(x_1, \cdots, x_n)dx_1\cdots dx_n,$$

where $w(x_1, \cdots, x_n)$ is the joint p.d. of the variables, and $H(R_1, R_2)$ the discontinuity factor of the region $R_1<\xi<R_2$. Upon writing $H(R_1, R_2)$ as a Fourier integral, one is led to a L.t. It should be noted that the two-sided L.t. is essentially the same as the ordinary characteristic function, but that the formalism of the Heaviside calculus is much more restricted than P. Lévy's inversion formula.

W. Feller.

Nakagami, Minoru and Ohno, Masami. On the resultant intensity of a number of vibrations whose phase are at random. Nippon Elec. Commun. Engrg., no. **19**, 129–137 (1940). [MF 4285]

The authors consider a random walk with unequal steps: there are n vibrations of constant amplitudes $l_1, \cdots, l_n$; the phases are independent random variables with rectangular distributions. The authors are interested in the expected intensity $E(l_1, \cdots, l_n)$ of the resultant vibration. Clearly

$$\pi E(l_1, l_2)=\int_0^\pi \{l_1^2+l_2^2-2l_1l_2\cos\theta\}^{\frac{1}{2}}d\theta,$$

which is essentially Legendre's complete integral. It is more convenient, however, to express $E(l_1, l_2)$ by means of Bessel functions. The case of n vibrations can then be treated by an inductive argument but leads to rather complicated expressions, which are studied by means of asymptotic series expansions.

W. Feller (Providence, R. I.).

Nakagami, Minoru. Study on the resultant amplitude of many vibrations whose phases and amplitudes are at random. Nippon Elec. Commun. Engrg., no. **22**, 69–92 (1940). [MF 4287]

The results of the paper reviewed above are generalized to the case where the amplitudes $l_1, \cdots, l_n$ are also random variables. However, in this paper not only the mean value of the resultant vibration, but also the probability distribution itself is computed [for the method of computation, cf. the second review above]. The complicated integrals involving Bessel functions are again expanded into asymptotic series, and approximative expressions for large n are obtained. Numerical examples are given.

W. Feller.

Labutin, D. N. On a generating function. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 139–147 (1940). (Russian) [MF 3308]

The generating functions occurring in the theory of Markoff's chains are investigated by means of the theory of matrices. Rather neat formal simplifications are obtained and an application to a probability problem is given.

M. Kac (Ithaca, N. Y.).

Hostinský, B. Sur le coefficient de corrélation. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 48–51. [MF 3501]

Let $\{X_n\}$ be a sequence of random variables forming a simple Markoff chain. Suppose that each X assumes the values $\alpha_1, \cdots, \alpha_r$ and that Prob. $\{X_k=\alpha_i\}=p_i^{(k)}$. The author proves that the correlation coefficient R_{ij} of X_i and X_j depends on $|i-j|$ alone if and only if $p_i^{(k)}$ does not depend on k.

M. Kac (Ithaca, N. Y.).

Savkevitch, V. Sur le schéma des urnes à composition variable. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 8–12 (1940). [MF 3590]

Suppose an urn U contains a white and b black balls. In a series of random drawings out of U the ball which was drawn is always replaced; in addition, $R>0$ new balls are added to U: if a white ball was drawn, ν white and $\rho=R-\nu\geqq 0$ black balls are added; if a black ball was drawn ν_1 black and $\rho_1=R-\nu_1$ white balls are added. The author computes the mean value and the standard deviation of the number X_n of white balls contained in U after n drawings. Moreover, he gives recurrence formulas for the third and fourth moments. In particular the symmetric case is considered, where $a=b$ and $\nu_1=\rho\neq 0$. It is shown that if $2(\nu-\nu_1)>R$ the variable H_n does not tend to a Gaussian distributed variable.

W. Feller (Providence, R. I.).

Bernstein, Serge. Sur un problème du schéma des urnes à composition variable. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 5–7 (1940). [MF 3589]

The author gives an independent direct proof of the

theorem mentioned at the end of the preceding review; it is also shown that if $2(\nu-\nu_1)<R$, the variable X_n is normally distributed in the limit. Using his general limit-theorems for dependent random variables, the author has previously [Bull. Acad. Sci. URSS [Izvestia] **4**, 137–150 (1940); cf. these Rev. **2**, 107] proved a slightly more general theorem.

W. Feller (Providence, R. I.).

Kitagawa, Tosio. The limit theorems of the stochastic contagious processes. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **1**, 167–194 (1941). [MF 3979]

Generalizing the Pólya-Eggenberger scheme (see below), the author treats chains of random variables where each term depends on all the previous ones. The definitions are the following ones: (i) A sequence of random variables $Y_1, \cdots, Y_n$ is said to be contagious if there exists a constant $\delta \geqq 0$ (the "parameter") and n generating distribution functions (g.d.f.) $F_k(x)$ such that the d.f. of Y_1 is $F_1(x)$, and the conditional d.f. of Y_{r+1} $(r=1, \cdots, n-1)$, when it is known that $Y_k=\alpha_k$ for $k=1, \cdots, r$ is given by

$$G_{r+1}(x; \alpha_1, \cdots, \alpha_r)=[1+r\delta]^{-1}\{F_{r+1}(x)+\delta \sum_{k=1}^{r} H(x-\alpha_k)\},$$

where $H(x)$ is the d.f. which equals 0 for $x<0$ and 1 for $x \geqq 0$. (ii) An infinite matrix $\|Y_{n,k}\|$ $(k=1, \cdots, m_n; n=1, 2, \cdots)$ defines a contagious process if for each n the sequence $Y_1, \cdots, Y_{m_n}$ is contagious in the above sense with a parameter δ_n and g.d.f. $F_{n,k}(x)$ $(k=1, \cdots, m_n)$. (iii) The process is called strong if $\delta_n \to d>0$, weak if $n\delta_n \to d \geqq 0$ and transitory if simultaneously $\delta_n \to 0$ and $n\delta_n \to \infty$.

Consider first a weak process with $m_n=n$, and assume that the g.d.f. $F_{n,k}(x)$ are step-functions with saltuses $\rho_{n,k}^{(\nu)}$, $\nu=0, \cdots, s$, at the points $0, \alpha_1, \cdots, \alpha_s$. Suppose furthermore that $\sum_{k=1}^{n} \rho_{n,k}^{(\nu)} \to h^{(\nu)}$ and that $\rho_{n,k}^{(\nu)} \to 0$ as $n \to \infty$, for every fixed $\nu \neq 0$ and uniformly with respect to $k=1, \cdots, m_n$. Then we have for the characteristic function $g_n(t)$ of $Y_{n,n}$

$$\lim_{n\to\infty} g_n(t)=\prod_{\nu=1}^{s} \{1-d(e^{it\alpha_\nu}-1)\}^{-h^{(\nu)}/d}.$$

This generalizes a result of Pólya and Eggenberger [Z. Angew. Math. Mech. **3**, 279–289 (1923)] to which it reduces if for each n the g.d.f. $F_{n,k}(x) \equiv F_n(x)$, where $F_n(x)$ has jumps at 0 and 1 with saltuses $\rho_n^{(0)}$ and $\rho_n^{(\nu)}=1-\rho_n^{(0)}$, respectively. If in the case of weak contagion all $F_{n,k}(x)$ coincide with a fixed step-function, the $Y_{n,n}$ are normally distributed in the limit. [Under some additional hypothesis on $F_{n,k}(x)$, the author also succeeds in determining the limiting d.f. in the case of strong contagion and of transition.]

W. Feller (Providence, R. I.).

***Lundberg, Ove. On Random Processes and their Application to Sickness and Accident Statistics.** University of Stockholm thesis, Uppsala, 1940. vii+172 pp.

Most of this paper (except for more expository parts) is devoted to special Markoff chains depending on a continuous time parameter t and with countably many states E_n, $n \geqq 0$. The author treats in particular the case where only the transition $E_n \to E_{n+1}$ is possible; this leads to the infinite system of differential equations $P_n'(t)=-p_n(t)P_n(t)+p_{n-1}(t)P_{n-1}(t)$, where $P_n(t)$ denotes the probability that at time t the state E_n is realized. Using the general methods of Feller [Math. Ann. **113**, 113–160 (1936); Trans. Amer. Math. Soc. **48**, 488–515 (1940); cf. these Rev. **2**, 101], the author investigates such systems in general and then treats two important special cases: (i) the compound Poisson process: it is obtained by considering an infinity of Poisson processes with a parameter x distributed according to a probability law $U(x)$. Obviously in this case

$$P_n(t)=\int_0^\infty e^{-tx}(tx)^n dU(x)/n!.$$

Several characteristic properties of such a process are deduced and the conditions are found under which a general process can be reduced to a compound Poisson process (for example, by introducing an operational time). Of special interest is the study of subprocesses, that is to say, of random samples out of the individual Poisson processes forming the compound Poisson process. Such a subprocess will, in general, be a contagious process (because, roughly speaking, the number of events in any time interval permits conclusions on the direction in which the empirical mean value of the x-parameter in the sample deviates from the mean value of $U(x)$). (ii) The Pólya process: it is obtained by a passage to the limit of the Pólya-Eggenberger model of urns [for reference cf. the preceding review]. A compound Poisson process is a Pólya process if $p_n(t)=(1+bn)/(1+bt)$. Several other characterizations are given. In a special chapter the moments, characteristic functions and limiting distributions are found as well as asymptotic series-expansions in the sense of Cramér. Finally interesting applications to accident and sickness insurance and statistics are given.

W. Feller (Providence, R. I.).

Ambarzumian, G. A. Study of a special case of a continuous stochastic process. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 120–138 (1940). (Russian) [MF 3307]

A particular case of the general theory of continuous stochastic processes (Kolmogoroff, Bernstein, Feller) is studied. This is of interest since it leads to the well-known Pearson distributions. *M. Kac* (Ithaca, N. Y.).

Doblin, V. Sur des mouvements mixtes. C. R. Acad. Sci. Paris **210**, 690–692 (1940). [MF 3915]

The author [=W. Doeblin] continues his researches on Markoff processes [C. R. Acad. Sci. Paris **207**, 705–707 (1938); **208**, 249–250 (1939); **210**, 365–367 (1940); these Rev. **1**, 343] analytically determined by a function $F(x, y, s, t)$: the probability that if a moving particle is in position x at time s it will be in position to the left of y at time t. It is supposed that F satisfies conditions giving a "mixed" process, in which the transitions proceed both continuously and by jumps. Proofs are not given. The distribution functions U of the continuous, and G of the discontinuous, parts of the motion are taken separately, and the mutual relations of F, U, G are found. U is determined as the solution of the differential equation of the continuous part of the motion [cf. Kolmogoroff, Math. Ann. **104**, 445–458 (1931)]. F finally appears as the sum of a series of multiple integrals of an increasing number of dimensions [cf. Feller, Math. Ann. **113**, 113–160 (1936); Trans. Amer. Math. Soc. **43**, 488–515 (1940); these Rev. **2**, 101]. *J. L. Doob* (Urbana, Ill.).

Yosida, Kôsaku and Kakutani, Shizuo. Operator-theoretical treatment of Markoff's process and mean ergodic theorem. Ann. of Math. (2) **42**, 188–228 (1941). [MF 3680]

The authors have treated temporally homogeneous Markoff processes, using the theory of linear operators in a Banach space, in a series of papers in the Proc. Imp. Acad. Tokyo of 1938, 1939 [cf. in part these Rev. **1**, 150, 343]. If $P(t, E)$ is the conditional probability that the moving

point t go into the set E in one transition, the operators concerned are defined by $x \to y$ ($x(E)$, $y(E)$ are completely additive set functions), where $y(E) = \int x(dt)P(t, E)$, and $\xi \to \eta$ (ξ, η are bounded measurable functions), where $\eta(t) = \int \xi(s)P(t, ds)$. They have found that hypotheses imposed by other authors [cf. the paper for full references] correspond to simple properties of the linear operators: complete continuity, etc. The present paper brings together all their results. As an illustrative example, the general theorems are applied to a process with a finite number of states, and the well-known results of the division of the states into ergodic groups, etc., are obtained. *J. L. Doob* (Urbana, Ill.).

Pitt, H. R. A special class of homogeneous random processes. J. London Math. Soc. **15**, 247–257 (1940). [MF 4035]

Let $f(x, t)$ for every value $t > 0$ be a distribution function in x, satisfying the Chapman-Kolmogoroff equation

$$f(x, t_1+t_2) = \int_{-\infty}^{+\infty} f(x-y, t_1)d_y f(y, t_2).$$

Generalizing a result of Wiener for the Gaussian distribution [Acta Math. **55**, 117–258 (1930), in particular § 12], it is shown that a Lebesgue measure in the space of real functions $\chi(t)$ can be defined in such a way that for any finite set of intervals $\{x_n, x_n'\}$ and any finite set of non-overlapping intervals $\{t_n, t_n'\}$ the set of functions $\chi(t)$ such that $\chi(t_n') - \chi(t_n) \subset I_n$ has the measure

$$\prod \{f(x_n', t_n'-t_n) - f(x_n, t_n'-t_n)\}.$$

Denote by $\phi(u, t)$ the characteristic function of $f(u, t)$, and by $\mathfrak{B}$ the set of terminating binary decimals. The main result of the paper consists in the proof that the following four statements are equivalent:

(i) $$f(x, t) = e^{-ct} \sum_{n=0}^{\infty} (ct)^n \psi_n(x)/n!,$$

where $\psi_n(x)$ is the n-fold convolution of a distribution function $\psi(x)$ with itself, and $\psi_0(x) = 1$. (ii) For arbitrarily small fixed t the values of $z = \phi(u, t)$ lie in a domain in which $\log z$ is regular and bounded. (iii) The set of functions $\chi(t)$ which vanish at all points of $\mathfrak{B}$ in a certain interval has positive measure. (iv) Almost all $\chi(t)$ are identical in $\mathfrak{B}$ with step functions with a finite number of discontinuities in any finite interval. *W. Feller* (Providence, R. I.).

Theoretical Statistics

*Peters, Charles C. and Van Voorhis, Walter R. **Statistical Procedures and Their Mathematical Bases.** McGraw-Hill Book Company, Inc., New York, 1940. viii+516 pp. $4.50.

Hartley, H. O. Recent advances in mathematical statistics: bibliography of mathematical statistics (1939). J. Roy. Statist. Soc. (N.S.) **103**, 534–560 (1940). [MF 3873]

Kendall, M. G. A recurrence relation for the semi-invariants of Pearson curves. Biometrika **32**, 81–82 (1941). [MF 3863]

From the differential equation of the Pearson curves $y(x)$ the author obtains easily an ordinary linear differential equation of second order for the moment generating function $M(t) = \int e^{tx} y(x)dx$; this equation can also be written as a non-linear equation of first order for the semi-invariant generating function $\log M(t)$. Writing the solutions in the form of power series one gets recurrence relations for the moments and for the semi-invariants, respectively. *W. Feller* (Providence, R. I.).

Kendall, M. G. Note on the distribution of quantiles for large samples. Suppl. J. Roy. Statist. Soc. **7**, 83–85 (1940). [MF 3898]

Suppose $x_1, x_2, \cdots, x_n$ are values of x (arranged in order of increasing magnitude) in a sample from a universe with a continuous distribution function $f(x)$. The author gives the joint distribution function of x_l and x_m ($l < m$), and from this he shows that, if l and m are taken as fixed proportions of n, then x_l and x_m are asymptotically (for large n) distributed according to a normal bivariate distribution, the variances and covariances of which are given. *S. S. Wilks* (Princeton, N. J.).

Bula, Clotilde A. Theory and evaluation of central moments in two dimensions. Sheppard's corrections. The simpler method of Mitropolsky. Revista Union Mat. Argentina **5**, 1–97 (1940) = Union Mat. Argentina, Publ. no. **9**, 97 pp. (1940). (Spanish) [MF 3998]

This is an expository account of joint moments of two statistical variables. The usual definitions are given of joint moments with respect to an arbitrary origin, and also with respect to the means of the two variables as the origin. The well-known Sheppard corrections for two-dimensional moments are given. The method of obtaining moments by the device of successive summation is discussed and illustrated with examples. *S. S. Wilks* (Princeton, N. J.).

Nicholson, C. A geometrical analysis of the frequency distribution of the ratio between two variables. Biometrika **32**, 16–28 (1941). [MF 3857]

The author derives the frequency curve of $(y+\bar{Y})/(x+\bar{X})$, where $\bar{X}$ and $\bar{Y}$ are constants and y and x are normally distributed and normally correlated random variables (variates). The solution of this problem which consists of a rather laborious evaluation of a certain double integral is due to R. C. Geary [J. Roy. Statist. Soc. **93** (1930)] and E. C. Fieler [Biometrica **24** (1932)]. The present author prefers to use the geometric language which, in the reviewer's opinion, does not contribute too much to the clarity of exposition. It is also unfortunate that the use of such expressions as "the angle KOA is α, which is the absolute value of the angle less than $\frac{1}{2}\pi$, etc." or "θ is the angular deviation of any plane from the angle α" makes the paper more difficult to read. *M. Kac* (Ithaca, N. Y.).

Quensel, Carl-Erik. Truncated normal curves and correlation distributions. Lunds Univ. Årsskrift (N.F.) **36**, no. 15, 17 pp. = Fysiogr. Sällskapets Handlingar (N.F.) **51**, no. 15, 17 pp. (1940). [MF 3905]

The author considers truncated normal distributions in one or two dimensions; in the latter case either one or both variates may be restricted to an interval (a, ∞). It is supposed that the primary material follows the normal distribution, but that it is restricted so as to include only individuals for which the attribute exceeds a certain minimum value. The problem is to determine the parameters of the total distribution. This is done by an adjustment of the moments and semi-invariants. The same problem in one dimension has been treated by Karl Pearson and Lee [Biometrika **6** (1908)]. *W. Feller* (Providence, R. I.).

Hotelling, Harold. Experimental determination of the maximum of a function. Ann. Math. Statistics **12**, 20–45 (1941). [MF 4002]

The author refers to several problems which require the finding of the value of x for which an unknown function $f(x)$ is a maximum or is a minimum. For this unknown function a polynomial is often used; but some caution is necessary. Before undertaking large-scale experiments, a series of preliminary experiments is advisable. These may assist in determining the degree of the polynomial to be used. The author discusses a great many situations that may arise, and gives detailed analysis for them. He also treats briefly unknown functions of two variables, and mentions some unsolved problems.

The first part of his Summary is as follows: "In determining the value ξ of x for which $f(x)$ is a maximum or minimum, a quadratic polynomial may be fitted to observations made for chosen values of x. The errors considered are of two kinds: sampling errors resulting from the inaccuracy in each observation, which diminish as the number of observations is increased, but increase if the values of x are chosen too close to the value sought; and biased errors resulting from the fact that $f(x)$ is not truly quadratic, which do not decrease when the number of observations increases with a fixed set of values of x, but do decrease when the deviations of x from the value sought are reduced."

E. L. Dodd (Austin, Tex.).

Thesen, Gudbrand. Über periodische Funktionen, ihre Observation und Ausgleichung. Skand. Aktuarietidskr. **1940**, 168–195 (1940). [MF 3932]

Given a function $f(x)$ with period p; that is, $f(x+p)=f(x)$. Now, observed $f(x_0)$, $f(x_0+\Delta)$, $f(x_0+2\Delta)$, $\cdots$ may exhibit a repetition-period, in general greater than p. The author discusses relations between periods and repetition-periods for the $f(x)$ above, and also for an $f(x)$, where $f(x+t_ip_i)=f(x)$, $i=1, 2, \cdots, n$; t_i an integer. An observation series is insufficient to determine p without further assumptions. The author assumes that $f(x)=A\phi(\alpha x+\beta)$. This makes possible several values for p connected by the equation $p_i=p_0\Delta/(\Delta+ip_i)$, $i=0, \pm1, \pm2, \cdots$. To determine p_0 more than one observation series is needed. The author applies his theory to a Norwegian mortality table. From the apparent periodicity in two graduations, he finds some evidence for a period $p_0=8$.

E. L. Dodd (Austin, Tex.).

Arley, Niels. On the distribution of relative errors from a normal population of errors. A discussion of some problems in the theory of errors. Danske Vid. Selsk. Math.-Fys. Medd. **18**, no. 3, 62 pp. (1940). [MF 3715]

Generalization of previous results of W. R. Thompson [Ann. Math. Statistics 6, 214 (1935)]. Let $x_1, x_2, \cdots, x_n$ be independent normal variates with their mean $\epsilon(x_i)=\sum_{j=1}^{m}a_{ij}p_j$, where the a's are known coefficients and the p's unknown parameters. Let the variance of x_i be $\sigma_i^2=\sigma^2/P_i$, where σ^2 is unknown but the weights P_i are known. Finally let ξ_i denote the estimate of $\epsilon(x_i)$ and S_i^2 the estimate of the variance of $x_i-\xi_i$, both obtained by the method of least squares. The author finds that then the distribution of $r_i=(x_i-\xi_i)/S_i$ is given by $p(r)=\text{const.}\,(n-m-r^2)^{(n-m-3)/2}$ for $|r|\leqq n-m$. This result is applied to give a criterion for rejecting divergent observations, as was done by Thompson. Further, the author suggests that the distribution of r may be used as a joint test for normality of a number of populations which have but small samples available. A similar idea was suggested recently by G. W. Brown [Ann. Math. Statistics **11**, 254–270 (1940); these Rev. **2**, 110].

J. Neyman.

Riebesell, P. Einige grundsätzliche Bemerkungen zur Frage des mittleren Fehlers. Mitt. Math. Ges. Hamburg **8**, part 2, 31–33 (1940). [MF 4115]

Cox, Gerald J. and Matuschak, Margaret C. An abbreviation of the method of least squares. J. Phys. Chem. **45**, 362–369 (1941). [MF 3992]

This paper concerns the numerical evaluation by least squares of the coefficients c_i in the equation of regression

$$y=c_0+c_1x+\cdots+c_mx^m$$

for the special case where the values of x are evenly spaced. Tables are given to facilitate the computation of the c_i's when $m<4$ and when there are fewer than 52 values of x. The use of the tables is then illustrated by an application to the weight-growth-curve of a rat. The paper is computational in nature.

T. E. Sterne (Ft. Bragg, N. C.).

Nörlund, N. E. Ausgleichung nach der Methode der kleinsten Quadrate bei gruppenweiser Anordnung der Beobachtungen. Acta Math. **72**, 283–353 (1940). [MF 3805]

In applications of least squares it is important not only to determine the most probable values of the unknowns $z_1, z_2, \cdots, z_n$, but also to find the mean errors of the z's or of functions of the z's. To compute these mean errors it is necessary to know the set of n^2 multipliers Q_{ij}, where the Q_{ij} are the elements of the matrix reciprocal to the quadratic matrix formed from the coefficients in the normal equations. The author considers special cases in which relatively simple explicit expressions can be obtained for the Q_{ij}, namely cases where the error equations are of the type

$$l_{is}+\lambda_{is}=k_i+z_{i+s-1}, \tag{1}$$

with $z_{n+s}=z_s$, or of the type

$$l_{is}+\lambda_{is}=k_i+\sum_{\nu=1}^{r}a_{s\nu}z_{i+\nu-1}, \tag{2}$$

with $z_{n+s}=z_s$. The l_{is} are observed quantities and the k's and z's are determined to minimize $\sum_{i=1}^{n}\sum_{s=1}^{m}\lambda_{is}^2$. For case (2) it is shown that

$$Q_{ij}=\frac{1}{n}\sum_{\nu=0}^{n-1}\frac{\cos(2\pi\nu/n)(i-j)}{c_r+2c_{r-1}\cos(2\pi\nu/n)+\cdots+2c_1\cos(2(r-1)\pi\nu/n)},$$

in which the c's are constants depending on the $a_{s,\nu}$. It is also shown that the Q_{ij} approach limits as n becomes infinite. Other expressions, not involving transcendental functions, are found for the Q_{ij}. Case (1) requires an auxiliary condition for the determination of the z's. If this be taken as $z_1+z_2+\cdots+z_n=0$, then (with a slight change of the author's notation)

$$Q_{ij}=\frac{m}{n}\sum_{\nu=1}^{n-1}\frac{\cos(2\pi\nu/n)(i-j)}{m^2-(\sin(\pi m\nu/n)/\sin(\pi\nu/n))^2}.$$

Here $m>1$, $n>2m-2$. The Q_{ij} may also be found from continued fractions, or by means of an interesting set of polynomials with integral coefficients.

W. E. Milne.

Deming, W. Edwards and Stephan, Frederick F. On a least squares adjustment of a sampled frequency table when the expected marginal totals are known. Ann. Math. Statistics **11**, 427–444 (1940). [MF 3421]

Suppose that in a census numeration of the values of

two characters for each individual only marginal totals are known for the whole population. To get information on the joint frequency of these characters, in a sample from the population a complete cross-tabulation is carried out so that in the two dimensional frequency table cell frequencies as well as marginal frequencies are known. The problem is now to write in the cell frequencies for the population in view of the sample frequencies in such a way as to preserve the population marginal frequencies. The first method given is that of a weighted least squares adjustment subject to fixed marginal totals. This is numerically illustrated for a two dimensional example and extended to the various cases arising in the enumeration of these characters. The second method is an iterative process based on the fact that if only one set of marginal totals is to be preserved the least squares method verifies that only proportionate adjustments need be made in the corresponding rows or columns. In the iterative process, first frequencies by rows are adjusted in this way and the table thus obtained is next adjusted by columns and if necessary this cycle is repeated. This process rapidly leads to stable cell frequencies which satisfy the requirement of fixed marginal totals and is readily extended to higher dimensional tables. It is illustrated in the same two dimensional example in which it gives the same results as the least squares method. There seems to be some implication that the two methods always lead to the same result, which does not seem to the reviewer to be the case since the one method depends on observed individual cell frequencies and the other on observed marginal totals only.

C. C. Craig (Ann Arbor, Mich.).

Eddington, A. S. The correction of statistics for accidental error. Monthly Not. Roy. Astr. Soc. **100**, 354–361 (1940). [MF 2355]

The author states: "I have given two formulae concerned with the elimination of the effects of accidental error of measurement in statistical investigations. The first derives the distribution function $v(x)$ of the true values of a characteristic x in terms of the distribution function $u(x)$ of the measured values of x; the second gives the mean true value $x-\bar{\epsilon}$ corresponding to a measured value x, so that an individual measure is improved by applying the correction $-\bar{\epsilon}$." The author then proceeds to give a more elaborate proof for these formulae than originally presented. In particular, he notes that "the improved values (taken all together) have a mean square error less than the mean square error of observation." As regards a certain danger of misapplying the first formula, discussed by Jeffreys, Eddington considers it unlikely that a practical investigator would be led here to false conclusions. On the contrary, the second formula is frequently employed in an illegitimate way, as pointed out by van Rhijn, in particular in connection with the parallaxes of stars. *E. L. Dodd* (Austin, Tex.).

Dwyer, P. S. The skewness of the residuals in linear regression theory. Ann. Math. Statistics **12**, 104–110 (1941). [MF 4008]

When b_0+b_1x is used as an estimate for y, there is a residual $\epsilon=y-b_0-b_1x$. The least square method is commonly used to determine b_0 and b_1. With barred letters to denote deviations from arithmetic means, the skewness of the residuals is expressed as

$$[\sum\bar{y}^3-3b_1\sum\bar{x}\bar{y}^2+3b_1^2\sum\bar{x}^2\bar{y}-b_1^3\sum\bar{x}^3]/N\sigma_\epsilon^3.$$

The author gives the details for the computation and applies his method to a correlation table for student grades.

E. L. Dodd (Austin, Tex.).

Johnson, Evan, Jr. Estimates of parameters by means of least squares. Ann. Math. Statistics **11**, 453–456 (1940). [MF 3424]

Let $x_1, x_2, \cdots, x_n$ denote observable random variables and F some function of these variables belonging to a class C. Further, let θ be a parameter involved in the probability law of the x's and I_F be the expectation of $(F-\theta)^2$. The function F_0, belonging to the class C for which I_{F_0} is a minimum is called the best estimate of θ relative to the class C. This conception differs from the familiar one of the best unbiased estimate in that it does not require that $\epsilon(F)\equiv\theta$. Occasionally, the determination of the best estimate requires knowledge of various parameters involved in the probability law of the x's. Assuming that all the x's are independent and follow the same normal law, the author shows that out of the class C_1 of functions $F=a\sum x_i$, with a arbitrary, the best estimate of $\xi=\epsilon(x_i)\neq 0$ is the one for which $a=(n+\sigma^2/\xi^2)^{-1}$. On the other hand, out of the class C_2 of functions $F=a\sum(x_i-\bar{x})^2$, the best estimate of σ^2 is perfectly determined by a knowledge of n and is obtained by putting $a=(n+1)^{-1}$. The first result, however, holds good whatever be the common probability law of the x's with finite $\xi\neq 0$ and σ. The second result requires the additional assumption $\beta_2=3$. *J. Neyman* (Berkeley, Calif.).

Dressel, Paul L. A symmetric method of obtaining unbiased estimates and expected values. Ann. Math. Statistics **12**, 84–90 (1941). [MF 4005]

It is shown that, because of the symmetry that exists between the value of a power product computed from a sample drawn without replacements from a finite population and its expected value, the problems of obtaining the expected values of moment functions in such samples and of finding unbiased estimates of given population moment functions are solved from the same set of equations. There are several illustrative examples, particularly for estimates and expected values of seminvariants of Thiele. It is shown also that the κ-system of seminvariants given by the author [Ann. Math. Statistics **11**, 53–57 (1940); these Rev. **1**, 249] and shown by him to be invariant under estimation in samples from an infinite population are likewise invariant if the samples come from a finite population.

C. C. Craig (Ann Arbor, Mich.).

Masuyama, Motosaburô. On the subdependency. Proc. Phys.-Math. Soc. Japan (3) **22**, 855–858 (1940). [MF 3256]

In the present paper the author examines singular cases which may arise in the theory developed in previous papers [the same Proc. **21**, 638–647 (1939) and **22**, 579–585 (1940); cf. these Rev. **1**, 151 and **2**, 110]. To explain the notation, we consider in an n-space two vector sets $\mathbf{f}_1(t)$, $\mathbf{f}_2(t)$, each set consisting of N vectors ($t=1, \cdots, N$); we may denote components by $(f_1(t))_i$. Then $[\mathbf{f}_1\mathbf{f}_2]$ is the $n\times n$ matrix with an i, j element equal to $\sum_{t=1}^{N}(f_1(t))_i(f_2(t))_j$. The sets $\mathbf{f}_1(t)$, $\mathbf{f}_2(t)$ are linearly totally dependent if tensors (that is, $n\times n$ matrices) $\mathbf{T}^1$, $\mathbf{T}^2$ exist independent of t such that $\mathbf{T}^1\cdot\mathbf{f}_1+\mathbf{T}^2\cdot\mathbf{f}_2=0$ for $t=1, \cdots, N$. Then it follows that

$$\mathbf{T}^1\cdot\{\mathbf{E}-\mathbf{R}(\mathbf{f}_1;\mathbf{f}_2)\}=0, \tag{1}$$

where $\mathbf{E}$ is the unit matrix and $\mathbf{R}(\mathbf{f}_1;\mathbf{f}_2)$ is the correlation tensor or matrix

$$\mathbf{R}(\mathbf{f}_1;\mathbf{f}_2)=[\mathbf{f}_1\mathbf{f}_2]\cdot[\mathbf{f}_2\mathbf{f}_2]^{-1}\cdot[\mathbf{f}_2\mathbf{f}_1]\cdot[\mathbf{f}_1\mathbf{f}_1]^{-1}.$$

If $\mathbf{T}^1$ is non-singular (that is, $\det\mathbf{T}^1\neq 0$), (1) implies $\mathbf{R}=\mathbf{E}$, and hence the correlation coefficient r^*, defined by $r^{*2}=\det\mathbf{R}$,

is unity. But a singular $\mathbf{T}^1$ will satisfy (1) if $\det(\mathbf{E}-\mathbf{R})=0$. Then the correlation coefficient will not be unity, although there is a linear dependence between $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$; in this case the two sets of vectors are said to be linearly subdependent. The argument is extended to the case where there are m sets of vectors instead of only two.

J. L. Synge (Toronto, Ont.).

Masuyama, Motosaburô. The variance tensor of vector set and a nature of the symmetric correlation coefficient. Proc. Phys.-Math. Soc. Japan (3) **22**, 858–861 (1940). [MF 3257]

In an n-space, let $\mathbf{U}(t)$ $(t=1, \cdots, N)$ be a set of N vectors; let $\mathbf{u}(t)$ be the deviation of $\mathbf{U}(t)$ from the mean. The author defines the variance tensor (or matrix) as $\overline{\mathbf{uu}}=[\mathbf{uu}]/N$, in the notation of the preceding review. He considers two sets of deviation vectors $\mathbf{g}(t)$, $\mathbf{h}(t)$ expressible in the form

$$\mathbf{g}=\mathbf{u}_1+\cdots+\mathbf{u}_p+\mathbf{v}_1+\cdots+\mathbf{v}_q,$$
$$\mathbf{h}=\mathbf{u}_1+\cdots+\mathbf{u}_p+\mathbf{w}_1+\cdots+\mathbf{w}_s,$$

where the sets on the right are orthogonal (that is, $[\mathbf{u}_1\mathbf{u}_2]=0$, $[\mathbf{u}_1\mathbf{v}_1]=0$, etc.). The correlation tensor of $\mathbf{g}$ and $\mathbf{h}$ can be expressed in terms of variances of the vectors on the right; in particular, if the $\mathbf{u}$'s, $\mathbf{v}$'s and $\mathbf{w}$'s have a common variance, the correlation tensor is the unit matrix multiplied by a constant. If further $s=q$, the square of the correlation coefficient is

$$r^{*2}(\mathbf{g};\mathbf{h})=\det \mathbf{R}(\mathbf{g};\mathbf{h})=p^{2n}/(p+q)^{2n}.$$

This gives a new meaning to the numerical value of r^*, which the author expresses as follows: ". . . the nth root of the absolute value of the symmetric correlation coefficient tends under some conditions to be the ratio of the number of causes common in the genesis of two vector sets to the whole number of independent causes on which each depends. . . ." *J. L. Synge* (Toronto, Ont.).

Horst, Paul. A non-graphical method for transforming an arbitrary factor matrix into a simple structure factor matrix. Psychometrika **6**, 79–99 (1941). [MF 4130]

The author lays down the principle that the simple structure matrix shall be such that for each factor the sum of squares of the significant loadings (in the non-technical sense) divided by the sum of squares of all the loadings shall be a maximum. An approximation method for solving the resulting equations which avoids graphical constructions is described. This rests on the assumptions that for a first approximation each factor has significant loadings in just two tests, that each test has a significant loading in not more than one factor, and that the number of factors in the simple structure matrix is equal to the number of columns in the original factor matrix that is to be reduced. The method is numerically illustrated in an example of Thurstone. *C. C. Craig* (Ann Arbor, Mich.).

Sitaraman, B. On correlation constants in mingled records. Math. Student **8**, 73–75 (1940). [MF 3282]

It is shown that, if the regression coefficient of y upon x is the same for each of k records separately, it is unaffected by mingling either if it is equal to the regression coefficient of the means of the k records or if the means of x in the k records are all equal. It is also shown that, if the intra-class correlation in each of the k records is positive, then the intra-class correlation of the mingled record must also be positive. But if the intra-class correlation in each record is negative, the intra-class correlation in the mingled record will be negative only under certain specified conditions.

W. A. Shewhart (New York, N. Y.).

Dwyer, P. S. The evaluation of multiple and partial correlation coefficients from the factorial matrix. Psychometrika **5**, 211–232 (1940). [MF 3157]

If in a linear regression problem the correlation matrix has been factor analyzed and the communality and uniqueness loadings obtained, the present paper shows how one may use the results in obtaining the various multiple and partial correlations and regression coefficients that may be desired. If the number of dependent variables is large and the number of common factors is small, the methods including the factorization will give the correlations and regression coefficients with considerably less labor than is required by more direct methods. The theory developed considerably overlaps that given by L. Guttman [cf. the following review] which was in press when this paper was received. But in the present paper actual computational methods and means of obtaining approximate results are emphasized and numerically illustrated. *C. C. Craig.*

Guttman, Louis. Multiple rectilinear prediction and the resolution into components. Psychometrika **5**, 75–99 (1940). [MF 3158]

The author begins with the usual assumption in factor analysis, namely, that each test in a battery of n tests is a linear function of r "common factors" $(r<n)$ and at most n "unique" factors, all common and unique factors being regarded as random variables which are mutually independent in the probability sense. A study is made of multiple and partial correlation coefficients and regression coefficients involved in this system of random variables. By using several well-known theorems in the algebra of matrices the author shows how to express these coefficients rather simply in terms of the factor loadings, that is, the coefficients of the common and unique factors in the linear functions defining the tests. In particular, he writes out the linear regression functions for predicting factor scores from test scores in terms of factor loadings. Incidentally, the factor loadings have to be estimated from observations, but the author does not discuss this problem. Finally the author considers limiting values approached by multiple and partial correlation coefficients as the number of tests increases indefinitely. These results are extensions of work previously done by Roff and Dwyer in Psychometrika.

S. S. Wilks (Princeton, N. J.).

Harman, Harry H. On the rectilinear prediction of oblique factors. Psychometrika **6**, 29–35 (1941). [MF 3793]

In a factor analysis in n variables in which there are $m(<n)$ common factors which are orthogonal, Lederman [Psychometrika **4**, 109–116 (1939)] and Guttman [cf. the paper reviewed above] have given a method for their estimation which requires the inverse of an $m\times m$ matrix instead of that of an $n\times n$ matrix. In this paper that method is generalized to the case of non-orthogonal factors and the general problem of estimation of factors is given a concise treatment. *C. C. Craig* (Ann Arbor, Mich.).

Hsu, P. L. On the limiting distribution of the canonical correlations. Biometrika **32**, 38–45 (1941). [MF 3859]

By methods similar to those used in his previous papers on roots of determinantal equations [Ann. Eugenics **9**, 250–258 (1939) and Proc. Edinburgh Math. Soc. (2) **6**, 185–189

(1940); cf. these Rev. 1, 248 and 2, 109] the author finds the limiting distributions of Hotelling's canonical correlations $r_1, r_2, \cdots, r_p$ in large samples, where $r_1, r_2, r_3, \cdots, r_p$ are the positive roots of the determinantal equation

$$\begin{vmatrix} \|ra_{ij}\| & \|b_{ik}\| \\ \|b_{gj}\| & \|rc_{gk}\| \end{vmatrix} = 0,$$

where $\|ra_{ij}\|$ $(i, j = 1, 2, \cdots, p)$, $\|b_{ik}\|$, $\|b_{gj}\|$ and $\|rc_{gk}\|$ $(g, h = 1, 2, \cdots, q)$, $p \leqq q$, are blocks of elements, where $a_{ij} = \sum_{t=1}^{n} x_{it}x_{jt}$, $b_{ik} = \sum_{t=1}^{n} x_{it}y_{kt}$, $c_{gk} = \sum_{t=1}^{n} y_{gt}y_{kt}$ $(n > p+q)$, and where the x_{it} and y_{gt} are sample values for $t = 1, 2, \cdots, n$ from a normal multivariate universe. The limiting distribution of the $r_1, r_2, \cdots, r_p$ is found under the assumption that the first μ_1 of the p population canonical correlations $\rho_1', \rho_2', \cdots, \rho_p'$ are each equal to ρ_1, say, the next μ_2 of them are equal to ρ_2 and so on, while the last $p-s$ of the population canonical correlations are zero, where $\rho_1 > \rho_2 > \cdots > \rho_\nu > 0$, $\mu_1 + \mu_2 + \cdots + \mu_\nu = s$. In particular, if $\eta_i = n^{\frac{1}{2}}(r_i - \rho_i')/(1 - \rho_i'^2)$, it is shown that the limiting joint distribution of the η_i is such that the first μ_1 η's, the next μ_2 η's and so on to the μ_ν η's in the νth set are independently distributed sets, having similar distributions of a form previously given by the author. The last $(p-s)$ η's are also distributed independently (in the limit) of the other η's and according to another form previously given by the author in his work on distribution of roots of determinantal equations. *S. S. Wilks.*

Bartlett, M. S. The statistical significance of canonical correlations. Biometrika **32**, 29–37 (1941). [MF 3858]

The recent results of Fisher and P. L. Hsu [Ann. Eugenics **9**, 238–249 and 250–258 (1939); these Rev. **1**, 248] on the simultaneous distribution of canonical correlations between vector variates in samples from multivariate normal populations is the point of departure. If all the population canonical correlations but one, say λ_1, are zero, we cannot be sure that the largest observed canonical correlation l_1 is associated with λ_1, so that l_1 is not a sufficient statistic for λ_1 and it is doubtful whether $p(l_1^2)$ or $p(l_1^2 | l_2^2, \cdots, l_p^2)$ gives the more powerful test for l_1. This point is illustrated in the simpler situation of two variates x_1 and x_2 distributed independently in rectangular distributions with means differing by α. If x_1 is the larger observed value, it is shown that though $p(x_1 | x_2)$ is not as powerful as $p(x_1)$ in detecting the shift α it is quite effective. The author then returns to the matter of determining significance levels for l_1 from $p(l_1^2 | l_2^2, \cdots, l_p^2)$. If two or more population canonical correlations are not zero the author recommends the use of an approximate test previously given by him [Proc. Cambridge Philos. Soc. **34**, 33–40 (1938)] which for this purpose is discussed and illustrated.

C. C. Craig (Ann Arbor, Mich.).

Woolley, Elliott B. The method of minimized areas as a basis for correlation analysis. Econometrica **9**, 38–62 (1941). [MF 3663]

It is proposed that linear regressions in two variables when both are subject to error be fitted by minimizing the sum of the residual areas formed by parallels to the coordinate axes through the observed points and the regression line. It is shown that the resulting regression of y on x is the same as that of x on y and the necessary formulae for the coefficients are developed. The expressions for the two variances of residuals and for the average of the area deviations are found. From this last an analog of the coefficient of correlation is defined which turns out to be identical with the ordinary coefficient of correlation. Other relations with the usual least squares method are also discussed.

C. C. Craig (Ann Arbor, Mich.).

Young, Gale. Maximum likelihood estimation and factor analysis. Psychometrika **6**, 49–53 (1941). [MF 3794]

In factor analysis an observed score matrix S may be regarded as arising from a true score matrix T (of lower rank) through the operation of independent random errors. Lawley has suggested [Proc. Roy. Soc. Edinburgh **60**, 64–82 (1940); these Rev. **2**, 110] estimating T from S by choosing the estimate which minimizes the probability density of occurrence of S from an arbitrary T. The procedure here differs from Eckart's "matrix approximation" by the appearance of weighting factors for the residuals. The solution is obtained by weighting S to get an S_1, finding the best unweighted approximation T_1 (of rank r) to S_1, and then by inverse weighting of T_1 to obtain T. This process, while reminiscent of the familiar "communalities" procedure, differs from this latter in that no change in direction of test vectors is introduced. As a consequence one source of underestimation of the number of factors is eliminated.

A. A. Bennett (Providence, R. I.).

Fisher, R. A. The precision of discriminant functions. Ann. Eugenics **10**, 422–429 (1940). [MF 3802]

In 1938 [Ann. Eugenics **7**, 179–188 (1938)] the author discussed a method of determining from a set of observed vectors of several components a linear "discriminant" function L which would produce a linear ordering of the given set of observed vectors with a maximum degree of stability in a sense equivalent to that in which the generalized variance of the observations taken about the principal axis associated with L would be minimized. In the present paper the problem of testing the significance of a given "discriminant" function as determined from observation against a hypothetical discriminant function is discussed and reduced to an analysis of variance. Unfortunately, many of the assumptions under which this significance test is valid are omitted. Several illustrative examples of the application of the significance test are given. *S. S. Wilks.*

Kendall, M. G. The derivation of multivariate sampling formulae from univariate formulae by symbolic operation. Ann. Eugenics **10**, 392–402 (1940). [MF 3801]

This paper is an extension to the multivariate case of methods presented in a recent paper by the same author [Ann. Eugenics **10**, 215–222 (1940); these Rev. **2**, 110] for determining sampling semi-invariants of Fisher's k-statistics. In fact the author reduces this extension to an application of certain symbolic operators on the sampling formulas of k-statistics in the univariate case. The method is illustrated on samples from a bivariate population. *S. S. Wilks.*

Kendall, M. G. On the method of maximum likelihood. J. Roy. Statist. Soc. (N.S.) **103**, 388–399 (1940). [MF 3200]

This paper is a non-technical expository type of discussion of the method of maximum likelihood, containing essentially nothing new. Five reasons proposed at one time or another by various writers for the use of the method are surveyed and commented upon. Included among these desiderata is the one dealing with the minimum variance properties of maximum likelihood estimates, when they exist, which were originally investigated by R. A. Fisher and subsequently

by others, properties which have now gained common acceptance as a sound basis for the validity of the method. Yet, somewhat surprisingly, the author rejects these properties as grounds for accepting the method, and tries to found his support of the method on a return to Bayes' theorem. *S. S. Wilks* (Princeton, N. J.).

Pearson, E. S. A note on further properties of statistical tests. Biometrika **32**, 59–61 (1941). [MF 3860]

Review of various setups of mathematical problems of statistical tests: λ criteria, most powerful tests, unbiased most powerful tests and the recent setup due to P. L. Hsu. This last test is the most powerful of those whose power function depends only on a given function of the parameters involved in the probability law of the observable variables. [See the following two reviews.] *J. Neyman.*

Simaika, J. B. On an optimum property of two important statistical tests. Biometrika **32**, 70–80 (1941). [MF 3862]

In two interesting theorems the author shows that Hotelling's T^2 test [Ann. Math. Statistics **2**, 359 (1931)] and the multiple correlation coefficient test have the optimum property in the sense of P. L. Hsu [see the following review]: out of all tests of the same hypotheses, with power functions depending on the same argument, these two tests are uniformly most powerful. *J. Neyman.*

Hsu, P. L. Analysis of variance from the power function standpoint. Biometrika **32**, 62–69 (1941). [MF 3861]

The importance of the author's results will be more easily appreciated in the light of the following circumstances. Consider the case where it is known that the n observable variables $x_1, x_2, \cdots, x_n$ are all mutually independent and normally distributed with the same unknown S.E. σ about the means $\epsilon(x_i)=\sum_{j=1}^{s}a_{ij}p_j$, where $s<n$, the a's are known coefficients and the p's unknown parameters. Denote by H the hypothesis that some $k<s$ linear functions $\theta_m=\sum_{j=1}^{s}b_{mj}p_j$ with known coefficients b_{mj} have some specified values θ_m^0. S. Kołodziejczyk [Biometrika **27**, 161 (1935)] has shown that, if $k>1$, then no uniformly most powerful test of the hypothesis H can exist, and has deduced a test based on the λ principle. P. C. Tang [Statist. Res. Mem. London **2**, 126 (1938)] has deduced and tabled the power function of the λ test of the hypothesis H. If θ_m' is the true value of the function θ_m then the power function appeared to depend only on $\varphi=\sum_{i=1}^{k}(\theta_i'-\theta_i^0)^2/\sigma^2$ and not directly on the true values of the particular parameters $p_1, p_2, \cdots, p_s$. The paper under review finds this important result: out of all tests of the hypothesis H whose power function depends on φ only, the λ test is uniformly most powerful. Apart from useful information on the test considered, the paper raises the general problem of tests of composite statistical hypotheses, whose power functions are independent of certain parameters involved in the probability law of the observable variables. A recent paper by G. B. Dantzig [Ann. Math. Statistics **11**, 186–192 (1940); these Rev. **1**, 348] shows that occasionally such tests do not exist. *J. Neyman.*

Hsu, C. T. On samples from a normal bivariate population. Ann. Math. Statistics **11**, 410–426 (1940). [MF 3420]

For samples from a normal bivariate population, the author considers tests of various hypotheses concerning the population parameters which are obtained by the method of Neyman and Pearson for deriving significance tests based on the maximum likelihood principle. By the use of a simple monotonic function of the Neyman-Pearson λ criterion and a rotation of the coordinate axes, the sampling distribution of each of the seven tests proposed is obtained, some of which the author observes are well known. In each case significance levels are readily obtained and a new table is provided for two of the tests. Three of the tests are on the hypothesis that $\rho=0$ and their power functions are discussed and tables are provided for numerical comparisons among them. It is verified that the tests, in case that the variances are equal or that in addition the means are equal, are increasingly more powerful than the test of $\rho=0$ with no assumptions concerning the values of the remaining parameters. *C. C. Craig* (Ann Arbor, Mich.).

van der Waerden, B. L. Biologische Konzentrationsauswertung. Ber. Verh. Sächs. Akad. Wiss. Leipzig **92**, 41–44 (1940). [MF 3488]

This paper deals with the biological assay problem in which it is desired to estimate, from a set of dosage-mortality data, the dosage required to produce a 50% mortality for a given organism. More specifically, if $h_1, h_2, \cdots, h_s$ are relative mortality frequencies in samples of size $n_1, n_2, \cdots, n_s$, respectively, exposed to dosage strengths having logarithms $x_1, x_2, \cdots, x_s$, respectively, the author proposes the following estimate of ξ, the log-dosage strength for 50% mortality:

$$L=\frac{x_s+x_{s+1}}{2}-\frac{1}{2}\sum_{1}^{s}(x_{j+1}-x_{j-1})h_j,$$

where the mortality corresponding to doses less than x_1 is "negligible" and corresponding to doses greater than x_s is "practically 100%." The variance of L is given, being a simple application of properties of the Bernoulli distribution. Considering the case where the x's are equally spaced with common difference δ, where the samples are of equal size n (large), and where the mortality function is the cumulative normal law with variance σ and mean ξ, the author finds the variance of L to be approximately $.564\delta\sigma/n$. The variance of the maximum likelihood estimate of ξ under the same assumptions is given as $.554\delta\sigma/n$. The efficiency (in R. A. Fisher's sense) claimed by the author for his method is thus approximately 554 : 564. *S. S. Wilks.*

Kermack, W. O. and M'Kendrick, A. G. The design and interpretation of experiments based on a four-fold table: the statistical assessment of the effect of treatment. Proc. Roy. Soc. Edinburgh **60**, 362–375 (1940). [MF 3363]

Krishna Iyer, P. V. The analysis of asymmetrical experiments with special reference to the partition of treatment sum of squares. Proc. Indian Acad. Sci., Sect. A. **11**, 369–375 (1940). [MF 2747]

The application of the principle of least squares to the analysis of nonsymmetrical field experiments, simple or complex, is here explained with special reference to splitting the total sum of squares for treatment into orthogonal components. The discussion is in the spirit of a principle announced by Yates, namely: "The general rule applicable to any group of fitted constants is to find the part of the sum of squares accounted for by fitting all the constants except those to be tested." The analysis of nonsymmetrical complex-experiments involving two variables is made by

finding (i) the sum of squares due to all the combinations that occur in the experiment and (ii) the sum of squares due to the mean effects by fitting constants. The sum of squares for interaction between the two variables is equal to (i) minus (ii). Applications are made to a number of simple randomized block patterns. *A. A. Bennett.*

Cornish, E. A. The analysis of covariance in quasi-factorial designs. Ann. Eugenics **10**, 269–279 (1940). [MF 3206]

Dodd, Edward L. The problem of assigning a length to the cycle to be found in a simple moving average and in a double moving average of chance data. Econometrica **9**, 25–37 (1941). [MF 3662]

In a previous paper [Ann. Math. Statistics **10**, 254–264 (1939); these Rev. **1**, 23] the author has defined the length of cycles resulting from the application of a graduation formula to a series of chance elements as the reciprocal of the probability of a certain feature of the graduated values. In this paper the author considers mainly the case when the graduation formula is a simple moving average. It is shown that in this case the graduated values have some features (frequent occurrence of insignificant maxima, etc.) which make it difficult to assign a cycle length according to the criteria considered in the author's previous paper. An empirical rule for the objective counting of the waves is set up and the author applies it to different chance series graduated by a simple moving average. The experimental results obtained seem to indicate that the average length of the waves produced is usually between $2.5n$ and $3.5n$, where n denotes the number of terms in the moving average.

A. Wald (New York, N. Y.).

Reiersøl, Olav. Confluence analysis by means of lag moments and other methods of confluence analysis. Econometrica **9**, 1–24 (1941). [MF 3661]

Here, as generally in the confluence analysis originated by Frisch, it is assumed that each of n statistical variates is the sum of two components, one of which is a systematic variate and the other a disturbance. The systematic parts are linearly related by a structural regression equation, and the disturbances are in general independent of each other and of the systematic parts. The main problem is to get as much information as possible concerning the vector whose components are coefficients in the structural regression equation from the matrix of covariances of the original n variates. If the variates are values in a time series they may be taken at times differing by τ and a lag-moment matrix set up of covariances between the variable x_i at time t and the variable x_j at the time $t+\tau$. Then under certain conditions of autouncorrelation of the disturbances with respect to the lag, it is shown that the structural regression may be determined from the Pfaffian of the skew symmetric matrix which is the difference between the lag-moment matrix and its transverse. If in empirical situations there is doubt of the applicability of this method the limits on the location of the structural regression vector with respect to elementary regressions obtained from the rows and columns of the adjoint of the lag-moment matrix are studied. This method is followed also in case the variables do not belong to a time series. In this second section use is made of the characteristic equation of the moment matrix and in particular various results are obtained in the case that there exist two more linear equations among the systematic parts.

C. C. Craig (Ann Arbor, Mich.).

Applications of the Theory of Probability

Kolmogoroff, A. N. On a new confirmation of Mendel's laws. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 37–41 (1940). [MF 3209]

Consider crossings of type $Aa \times Aa$, where the character A is dominant over a. According to Mendelian laws it should be expected that one fourth of the total population will show the character a (be of the aa type). The author computes the probable deviations from this expected average. He then investigates the empirical data which have recently been produced by different authors in order to refute the Mendelian concepts. It is found that these data, upon a correct analysis, confirm the Mendelian laws.

W. Feller (Providence, R. I.).

Etherington, I. M. H. Non-associative algebra and the symbolism of genetics. Proc. Roy. Soc. Edinburgh. Sect. B. **61**, 24–42 (1941). [MF 4141]

Many problems of genetics (for example, frequency distributions of genes and zygotes in the progeny of a couple) can be treated by means of special algebras if the sign $\times$ of genetical multiplication is interpreted literally as the sign of multiplication. In previous papers [Proc. Roy. Soc. Edinburgh **59**, 242–258 (1939); J. London Math. Soc. **15**, 136–149 (1940); cf. these Rev. **1**, 99 and **2**, 121] the author has investigated these algebras. In the present paper the symbolism is considered more from the biologist's point of view and exemplified by the treatment of some simple special cases. No mathematical progress is claimed.

W. Feller (Providence, R. I.).

Kostitzin, V. A. Sur la ségrégation physiologique et la variation des especes. Acta Bioth. Ser. A. **5**, 160–168 (1940). [MF 2860]

The author studies the coexistence of two species which are similar in all respects except as regards the age of maturity or ability to reproduce. The assumptions made concerning the forces determining the rates of increase, etc., are similar to those in the Volterra-Lotka theory of struggle for life. The author studies the development of the sizes of the populations from given initial conditions. The investigation is made in order to examine the compatibility of the coexistence of species with the usual conceptions of natural selection. *W. Feller* (Providence, R. I.).

Kostitzin, V. A. Sur la loi logistique et ses généralisations. Acta Bioth. Ser. A. **5**, 155–159 (1940). [MF 2859]

The author makes critical observations upon the law of population growth which he regards as improperly called the logistic law. The cases in which the vital coefficients instead of being sensibly constant are variable and in particular are periodic are discussed. Only non-systematic variations in the coefficients leave the problem essentially invariant. An interesting case in which a population requires a certain initial minimum for mere survival, but in which, for any initial census greater than this minimum, the eventual distribution becomes independent of the initial number, is treated briefly. *A. A. Bennett.*

Lotka, Alfred J. The theory of industrial replacement. A commentary. Skand. Aktuarietidskr. **1940**, 1–14 (1940). [MF 2961]

A rejoinder to Preinreich's recent criticism of Lotka's

methods [Skand. Aktuarietidskr. **1939**, 1–9 (1939); these Rev. **1**, 154].

Rhodes, E. C. Population mathematics. II and III. J. Roy. Statist. Soc. (N.S.) **103**, 218–245, 362–387 (1940). [MF 3062, 3199]

Continuation of the investigations in the same volume, pp. 61–89 [cf. these Rev. **1**, 250]. The present papers are concerned mostly with special cases, numerical computations and fittings to empirical data; special attention is paid to Lotka's method for determining the roots of the so-called characteristic equation. In an appendix the author considers different types of growth-functions. *W. Feller.*

Hadwiger, H. Eine analytische Reproduktionsfunktion für biologische Gesamtheiten. Skand. Aktuarietidskr. **1940**, 101–113 (1940). [MF 3929]

Consider the progeny of a single individual born at $t=0$. Let $\phi_n(x)$ represent the density of individuals reproduced at time x by individuals belonging to the nth generation, so that

$$(*) \qquad \phi_{n+1}(x)=\int_0^t \phi_n(x-y)\phi_1(y)dy.$$

The author assumes (without explanation) that $\phi_n(x)$ is of the form $\psi(x, na)$, where $a>0$ is a parameter. Equation (*) then reduces to a well-known integral relation for $\psi(x, a)$, which admits of the solution

$$\psi(x, a)=a\pi^{-1}x^{-1}\exp\{Ca-\Lambda x-a^2x^{-1}\}.$$

The author verifies this by direct computation and claims to have found the analytic expression for $\phi_n(x)$, though in reality there are other solutions of the same equation. Some simple remarks about the total number of regenerations follow. *W. Feller* (Providence, R. I.).

Brown, A. W. A note on the use of a Pearson type III function in renewal theory. Ann. Math. Statistics **11**, 448–453 (1940). [MF 3423]

The paper refers to the practical solution of the integral equation

$$B(t)=B_1(t)+\int_0^t B(t-x)f(x)dx,$$

when $f(x)$ is a function of Pearson type III ($Ax^k e^{-\alpha x}$). [Cf. also Hadwiger and Ruchti, Metron **13**, no. 4, 17–26 (1939); these Rev. **1**, 348.] *W. Feller* (Providence, R. I.).

Kitagawa, T. and Huruya, S. The application of the limit theorem of the contagious stochastic process to the contagious diseases. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **1**, 195–207 (1941). [MF 3980]

Statistics on the number of deaths resulting from some diseases in a Japanese prefecture are compared with the Poisson law and the Pólya-Eggenberger law [Z. Angew. Math. Mech. **3**, 276–289 (1923)]. Cf. also a Stockholm thesis by O. Lundberg, these Rev. **2**, 230. *W. Feller.*

Seal, H. L. Tests of a mortality table graduation. J. Inst. Actuar. **71**, 5–48, discussion 48–67 (1941). [MF 4139]

Exposition of the methods used for the problem mentioned in the title. Pages 48–67 contain a discussion of the paper contributed by various authors.

Huntington, Edward V. Methods of Apportionment in Congress. A survey of methods of apportionment in Congress. Senate document no. **304**, 76th Congress, 3d Session. United States Government Printing Office, Washington, 1940. v+41 pp. [MF 3174]

NUMERICAL AND GRAPHICAL METHODS

***Glaisher, J. W. L., Bickley, W. G., Gwyther, C. E., Miller, J. C. P. and Ternouth, E. J. Table of Powers Giving Integral Powers of Integers.** British Association for the Advancement of Science, Mathematical Tables, vol. 9. Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1940. xii+132 pp. $4.25.

This table gives a little more than 18000 values of the function x^n for positive integral values of x and n as follows: $n=2, 3, \cdots, 12$, $x<1100$; $n=13, 14, \cdots, 20$, $x<300$; $n=21, 22, \cdots, 50$, $x\leqq 120$. Powers above the 30th, except the 40th and 50th, are not printed in full, only the first 21 to 26 digits being given. The arrangement of the table is very uniform and convenient considering the great differences in the sizes of the entries. Each opening of the table presents values of x^n for 100 consecutive values of x and for two or more values of n. If x is fixed, values of x^n for consecutive values of n usually occur on the same line of consecutive openings or columns.

This volume is based on a table of the first dozen powers of the first 1000 integers computed under the direction of J. W. L. Glaisher about 70 years ago, set in type by some unknown printer, but never published. A copy of page proof, perhaps the only one in existence, was subjected to careful checking and found to be free from error, except for an obvious missing digit. In connection with the checking of the present volume the coefficients of the polynomial in X which represents the power sum $\sum_{x=1}^{X} x^n$ were completed as far as $n=50$. These are tabulated at the end of the volume together with a one page table of the binomial coefficients $\binom{n}{k}$ for $n\leqq 50$, $k\leqq 10$. The introduction contains a bibliography of 11 tables of powers with the errata found on comparing them with the present volume.

D. H. Lehmer (Berkeley, Calif.).

***Allen, Edward S. Six-Place Tables.** 6th ed. McGraw-Hill Book Company, Inc., New York, 1941. xxiii+181 pp. $1.50.

A selection of tables of squares, cubes, square roots, cube roots, fifth roots and powers, circumferences and areas of circles, common logarithms of numbers and of the trigonometric functions, the natural trigonometric functions, natural logarithms, exponential and hyperbolic functions, and integrals.

***Peters, Jean. Siebenstellige Logarithmentafel. Bd. 1. Logarithmen der Zahlen, Antilogarithmen, Additions- und Subtraktionslogarithmen, nebst e.** Reichsamt für Landesaufnahme, Berlin, 1940. vii+493 pp. RM 8.

***Tables of Sine, Cosine and Exponential Integrals. Vol. I.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 765-97-3-10; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1940. xxvi+444 pp. $2.00.

The functions tabulated are

$$\mathrm{Si}(x)=\int_0^x (\sin t)dt/t, \qquad \mathrm{Ci}(x)=\int_\infty^x (\cos t)dt/t,$$

$$\mathrm{Ei}(x)=\int_{-\infty}^x e^t dt/t, \qquad -\mathrm{Ei}(-x)=\int_x^\infty e^{-t}dt/t.$$

In Table I (the principal table of the volume, 437 pp.), these functions are tabulated to 9 decimals over the range from 0 to 2 at intervals of 0.0001. This interval was chosen so that interpolation by formulae involving second central differences will yield the fullest attainable accuracy over most of the range. The second central differences (when significant) are also tabulated to facilitate interpolation. In the range between 0 and 0.01, where interpolation for the functions $\mathrm{Ci}(x)$, $\mathrm{Ei}(x)$ and $-\mathrm{Ei}(-x)$ is bad (these functions become logarithmically infinite at $x=0$), the functions $\mathrm{Ci}(x)-\log_e x$, $\mathrm{Ei}(x)-\log_e x$ and $-\mathrm{Ci}(-x)+\log_e x$ are also tabulated. Table II gives the four functions to nine decimal places from 0 to 10 at intervals of 0.1. Tables III and IV give $\frac{1}{2}p(1-p^2)$ and $p(1-p)$, respectively, for use in interpolation formulas using second differences. The introduction describes the tables and the method of computation, gives asymptotic formulas for use beyond the limit of the tables, describes suitable methods of direct and inverse interpolation, gives the graphs of the four functions and provides a bibliography of the functions and their uses. The book is an excellent piece of work. *W. E. Milne.*

Vandrey, F. Tafel der acht ersten Kugelfunktionen zweiter Art. Z. Angew. Math. Mech. **20**, 277–279 (1940).

The tables are given to five decimal places and cover the interval $0\leqq x\leqq 1$ in steps of .01.

Heuman, Carl. Tables of complete elliptic integrals. J. Math. Phys. Mass. Inst. Tech. **20**, 127–206 (1941). [MF 4126]

In certain dynamical problems the desired quantities are expressible in terms of Legendre's complete elliptic integrals of the first, second and third kinds. The tables here given are designed to facilitate such computations by giving the complete integrals multiplied by certain factors which are found with them in these problems. The modified functions of the first and second kind, $F_0(\alpha)$ and $E_0(\alpha)$, are given for each value of α between 0 and 90°, by tenths of a degree, to six decimal places. Since the differences of $F_0(\alpha)$ become large when $\alpha>65°$, an auxiliary function $G_0(\alpha)$ is tabulated for α from 65° to 90°, from which $F_0(\alpha)$ may be easily computed for intermediate values. For the dynamical applications, the parameter which occurs in the integral of the third kind may be taken as real and in a restricted range, and so replaced by a parametric angle β. This leads to the modified function of the third kind $\Lambda_0(\alpha, \beta)$, which is tabulated for α from 0 to 90° and β from 0 to 90°, by even degrees, to six decimal places. Methods of interpolating in the table are described, and a small auxiliary table of use in certain ranges is given. In calculating these tables from those of Legendre the author discovered some 40 misprints which are listed as an errata sheet either for Legendre's original tables or some recent photographic reproductions. In an appendix some specific dynamical problems for which the tables can be used are discussed, and references given to the author's earlier publications in Swedish where further details can be found. *P. Franklin* (Cambridge, Mass.).

Fletcher, Alan. A table of complete elliptic integrals. Philos. Mag. (7) **30**, 516–519 (1940). [MF 3711]

This table was made in connection with work on Laplace coefficients in dynamical astronomy [see Monthly Not. Roy. Astr. Soc. **99**, 259 (1939)]. On this account, it tabulates the values of the complete elliptic integrals E and K directly in terms of the modulus k as argument, in preference to using k^2 or $\sin^{-1} k$. Values of k, $0\leqq k\leqq 1$, are taken at intervals of 0.01 and the corresponding values of E, K and M are given to ten decimals. M is the Gaussian arithmetic-geometric mean of 1 and the complementary modulus k'. *M. A. Basoco* (Lincoln, Neb.).

Kotani, Masao and Amemiya, Ayao. Tables of integrals useful for the calculations of molecular energies. II. Proc. Phys.-Math. Soc. Japan (3) **22**, extra no. pp. 1–28 (1940). [MF 3065]

The first part appeared as Extra Number 1 [70 pp.] to vol. **20** (1938) of the same Proceedings.

Dwight, H. B. Values of the Bessel functions ber x **and** bei x **and their derivatives.** Trans. Amer. Inst. Elec. Engrs. **58**, 787–790 (1939). [MF 3536]

The present paper gives the values of ber x and bei x and of their derivatives of first order for the range $0<x<20$ at intervals of 0.1. The corresponding absolute values and the arguments of ber $x+i$ bei $x=J_0(xi^{3/2})$ are also tabulated.

Lowan, Arnold N. and Blanch, Gertrude. Errors in Hayashi's table of Bessel functions for complex arguments. Bull. Amer. Math. Soc. **47**, 291–293 (1941). [MF 4179]

Brixy, Eduard. Einige Integrale der Besselschen Funktionen reellen positiven Argumentes. Z. Angew. Math. Mech. **20**, 236–238 (1940).

Numerical values are given for some special integrals.

Neuschuler, L. Sur les tables optimales à trois membres des fonctions de deux variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 843–846 (1939). [MF 2061]

The author considers the problem of replacing an ordinary double entry table of a function $f(x, y)$, too extensive to be feasible, by a set of tables of functions $\varphi_1(t)$, $\varphi_2(t)$, $F(t)$ in terms of which $f(x, y)$ may be simply expressed, as for example: $f(x, y)=F[\phi_1(x)+\phi_2(y)]$. In an attempt to reduce the number of entries still further, a series of values $z_1, z_2, \cdots, z_n$ are selected and for each z_i are tabulated the functions $\psi_1(y)=\varphi_1(z_i)+\varphi_2(y)$, $\psi_2(x)=\varphi_1(x)-\phi_1(z_i)$ $(z_i\leqq x\leqq z_{i+1})$, from which $f(x, y)=F[\psi_1(y)+\psi_2(x)]$. No actual example of the effectiveness of the method is given. *D. H. Lehmer* (Berkeley, Calif.).

Hancock, R. T. The analysis of compound exponential curves. Engineer **168**, 492 (1939). [MF 3537]

The author sketches a method for fitting curves $x=ae^{bt}+ce^{dt}$ to empirical data. *W. Feller* (Providence, R. I.).

Bickley, W. G. Formulae for numerical differentiation. Math. Gaz. **25**, 19–27 (1941). [MF 3927]

The paper presents tables for the calculation of the derivatives $D^m y_p$ of a function $y(x)$ in terms of the ordinates y_p at $n+1$ equally spaced points $x_p = x_0 + ph$. These tables give the numerical values of ${}_{mn}A_{pr}$ in the formula

$$h^m D^m y_p / m! = (1/n!) \sum_{r=0}^{n} {}_{mn}A_{pr} y_r + {}_{mn}E_p,$$

together with bounds for the error term ${}_{mn}E_p$. The scope of the tables is $n = 2, 3, 4, 5, 6$, m from 1 to n, p from 1 to n; $n = 8, 10$, m from 1 to 4, p from 1 to n. At the top of page 23, A_8 is misprinted for A_5. The use of the tables in the numerical solution of a differential equation is illustrated by an example. *W. E. Milne* (Corvallis, Ore.).

Gutenmacher, L. Künstliche elektrische Modelle vieldimensionaler Körper. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 198–202 (1940). [MF 3215]

The partial difference equation corresponding to the Laplace equation can, in a well-known manner, be represented by electrical potentials or currents in a simple network consisting of homogeneous wires. In the present note a method is indicated whereby a great variety of partial difference equations of the second order can be realized by electrical models. The principle is to connect the netpoints in the simple net with more complicated units each of which may contain resistance, inductivity and capacity. By selecting these units suitably, we force an electrical current in our network to represent approximate solutions of general classes of partial differential equations. If proper boundary or initial conditions are secured, we obtain a method for a practical solution of corresponding boundary or initial value problems. The author reports that in different non-trivial cases actual experiments have yielded results not more than 2% different from the theoretical values. *R. Courant.*

Nyström, E. J. Graphisch-mechanische Auswertung von Doppelintegralen, insbesondere bei Oberflächenbestimmungen. Soc. Sci. Fenn. Comment. Phys.-Math. **10**, no. 8, 12 pp. (1940). [MF 3994]

A method is described for finding the area of a surface, that is, for evaluating the integral $\int\int \sec\gamma dxdy$, where γ is the angle that the normal to the surface element makes with the vertical axis. The surface is assumed to be given graphically by its level curves with constant differences of elevation between levels. The value of $\sec\gamma$ is determined graphically at a set of lattice points by means of a transparent piece of celluloid with rectangular rulings. Using these values as new ordinates, the volume of the solid thus formed is approximated by one of the usual methods. *P. W. Ketchum* (Urbana, Ill.).

Laurila, E. Über das Nyströmsche Stieltjesplanimeter. Soc. Sci. Fenn. Comment. Phys.-Math. **10**, no. 7, 19 pp. (1940). [MF 3993]

A Stieltjes planimeter is an instrument for evaluating integrals of Stieltjes form

$$\int_{t_1}^{t_2} f(t)dh(t) = \int_{t_1}^{t_2} f(t)h'(t)dt,$$

where $f(t)$ and $h(t)$ are given functions. A design is given for an instrument of this kind in which t_2 is variable, so as to give an indefinite integral. An ordinary revolving table planimeter is used in such a way that the increment of rotation of the table is proportional to $dh(t)$. Instead of constructing this device, the author made an attachment for a Mader-Ott analyzer which accomplishes a similar result except that t_2 is fixed. The curve for $h(t)$ is on a drawing board which is supported by and moves with one of the carriages of the analyzer. The curve may be followed either manually or automatically by means of a template. Various important applications are given including evaluation of integrals of the form $\int t^n f(t)dt$, $\int F(y(x))dx$, $\int\int x^n dxdy$, $\int y^3 dx$, $\int y^{\frac{1}{2}} dx$, etc. *P. W. Ketchum* (Urbana, Ill.).

Laurila, Erkki. Zur mechanischen Berechnung des Fourierschen Integrals. Soc. Sci. Fenn. Comment. Phys.-Math. **10**, no. 15, 16 pp. (1940). [MF 3995]

The author shows how any harmonic analyzer can be used to evaluate the Fourier transform

$$F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) e^{itx} dx$$

of a given real valued function $f(x)$. The infinite limits are first replaced by a finite range $(-R, R)$ large enough to make the error negligible. If real and imaginary parts are separated after setting $F(x) = U(x) + iV(x)$, then it will be seen that the coefficients a_n and b_n in the Fourier expansion of $f(x)$ in the interval $(-R, R)$ are proportional to the values of $U(x)$ and $V(x)$, respectively, at the points $\pi n/R$. Thus, an analyzer which gives a certain number of Fourier coefficients will give at the same time a corresponding number of equally spaced values of the Fourier transform. The theory and use for this purpose of the Mader-Ott analyzer is discussed in detail. *P. W. Ketchum* (Urbana, Ill.).

Tucker, Ledyard R. A matrix multiplier. Psychometrika **5**, 289–294 (1940). [MF 3318]

A machine to expedite matrix multiplication has been obtained by modifying the International Test Scoring Machine. In this device the values of the elements of product matrices are indicated one at a time by a milliammeter. It is claimed that, with reasonable care, 2-place accuracy can be obtained. Operating procedure and time estimates are given. *G. R. Stibitz* (New York, N. Y.).

Walther, A. Neuzeitliche mathematische Maschinen. Elektrotech. Z. **61**, 33–36 (1940). [MF 4092]

Brown, S. Leroy and Wheeler, Lisle L. A mechanical method for graphical solution of polynomials. J. Franklin Inst. **231**, 223–243 (1941). [MF 3904]

The use of a harmonic synthesizer for the extraction of complex roots of polynomials is illustrated. The methods employed were proposed by Fry and Kempner [Bull. Amer. Math. Soc. **41**, 809–843 (1935)]. Graphs revealing real roots only are also shown. *R. L. Dietzold* (New York, N. Y.).

Nikolaev, P. V. Polynômes de Masseau et transformations rationnelles de nomogrammes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 582–584 (1940). [MF 3611]

Nikolaev, P. V. L'anamorphose des polynômes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 774–777 (1940). [MF 3616]

The author applies methods of algebraic geometry to the study of those polynomials $F(t_1, t_2, t_3)$ which can be represented by rational nomograms. Such a "Masseau" polynomial must be identically equal to a determinant of the form $|f_{ij}(t_j)|$, where the scale functions f_{ij}, $i, j = 1, 2, 3$, are rational. In the first paper it is shown that with certain exceptions all nomograms of a given Masseau polynomial are collinear. In the second paper are criteria for determining if a given polynomial is a Masseau polynomial and an algorithm for the construction of the functions f_{ij} when they exist. *P. W. Ketchum* (Urbana, Ill.).

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